Conditions for similarity to a normal operator involving resolvent estimates

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Summary

1. Pointwise resolvent estimates

2. Mean squares resolvent estimates

3. Tools used in the proofs

Summary

Previous results and main result

- T a bounded operator on a Hilbert space H. $\sigma(T)$ contained in a smooth curve Γ without self-intersections.
- Stampfli (1965): If $\|(T-\lambda)^{-1}\| \leq \operatorname{dist}(\lambda,\Gamma)^{-1}$ in some neighbourhood of Γ, then T is normal.
- First proved by Nieminen (1962) for $\Gamma = \mathbb{R}$ and Donoghue (1963) for $\Gamma = \mathbb{T}$.
- Natural question: What about similarity to a normal operator?
- There are operators T with $\sigma(T) \subset \Gamma$ and such that $\|(T-\lambda)^{-1}\| \leq C \operatorname{dist}(\lambda,\Gamma)^{-1}$ but T is not similar to a normal. Benamara-Nikolski (1999), Nikolski-Treil (2002).

Theorem A

If Ω is a $C^{1+\alpha}$ domain, $\Gamma = \partial\Omega$, $\sigma(T) \subset \Gamma$, U a neighbourhood of Γ,

$$\|(T-\lambda)^{-1}\| \leq \operatorname{dist}(\lambda,\Gamma)^{-1}, \quad \lambda \in U \setminus \overline{\Omega},$$

$$\|(T-\lambda)^{-1}\| \leq C \operatorname{dist}(\lambda,\Gamma)^{-1}, \quad \lambda \in \Omega,$$

then T is similar to a normal.

The conditions of the theorem can be interchanged (constant 1 inside and constant C outside).

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Some examples

- If T is a contraction, then $\|(T-\lambda)^{-1}\| \leq (|\lambda|-1)^{-1}$ for $|\lambda| > 1$.
- A contraction T is similar to a unitary if and only if $\|\Theta_T(z)^{-1}\| \leq C$ for $|z| < 1$ (Sz.Nagy-Foias).
- A contraction similar to a unitary satisfies the hypotheses of Theorem A. It can be transplanted to other domains by a Riemann mapping.
- ρ-contractions also serve as examples. T is a ρ-contraction if there is a unitary U on a larger space K such that $T^n = \rho P_H U^n | H$ for $n \geq 1$.
- If T is a ρ-contraction with $\rho \geq 2$, then $\|(T-\lambda)^{-1}\| \leq (|\lambda|-1)^{-1}$ for $1 < |\lambda| < (\rho-1)/(\rho-2)$.
- A ρ-contraction similar to a unitary satisfies the hypotheses of Theorem A.
- An example of a 2-contraction similar to a unitary which is not a contraction: The bilateral weighted shift in $\ell^2(\mathbb{Z})$ with weights $\ldots, 1, 1, \alpha, \beta, 1, 1, \ldots$, where $\alpha, \beta > 0$, $\max(\alpha, \beta) > 1$ and $\alpha^2 + \beta^2 < 2$.

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Summary

Conditions for similarity to a unitary

Consider the following conditions:

- Sim(T), T is similar to a unitary.
- PowBd(T), T is power bounded ($\|T^n\| \leq C$, $n \geq 0$).
- LMean^{2}RG(T), T satisfies

$$\int_{|\lambda|=r} \|(T-\lambda)^{-1}x\|^2 \, |d\lambda| \leq C\|x\|^2(r-1)^{-1},$$

  for every $r > 1$ and $x \in H$.
- LRG(T), T satisfies $\|(T-\lambda)^{-1}\| \leq C(|\lambda|-1)^{-1}$, for every $|\lambda| > 1$.

Then Sim(T) $\Rightarrow$ PowBd(T) $\Rightarrow$ LMean^{2}RG(T) $\Rightarrow$ LRG(T).

- PowBd(T) & PowBd(T^{-1}) $\Rightarrow$ Sim(T) (Sz. Nagy, 1947).
- PowBd(T) & LRG(T^{-1}) $\Rightarrow$ Sim(T) (van Casteren, 1980).
- LMean^{2}RG(T) & LMean^{2}RG(T^*) & LRG(T^{-1}) $\Rightarrow$ Sim(T) (van Casteren, 1983).
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LMean^{2}RG for Jordan curves

- Ω a Jordan domain, $\Gamma = \partial\Omega$.
- $\{\gamma_s\}_{0<s<1}$ a collection of Jordan curves tends nicely to Γ if:
  1. $C^{-1}s \leq \mathrm{dist}(x, \Gamma) \leq Cs, \quad x \in \gamma_s$
  2. $\mathrm{length}(\gamma_s \cap B(x, r)) \leq Cr$
- If $\gamma_s \subset \Omega$ ($\gamma_s \subset \mathbb{C} \setminus \overline{\Omega}$) for all s, we say that $\{\gamma_s\}$ tends to Γ from the inside (outside).

We consider the condition

$$\int_{\gamma_s} \|(T-\lambda)^{-1}x\|^2 \, |d\lambda| \leq C\|x\|^2 s^{-1},$$

where $\{\gamma_s\}_{0<s<1}$ tends to Γ from the inside/outside. This condition does not depend on the particular choice of $\{\gamma_s\}$.

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Generalization of van Casteren (1983)

van Casteren (for $\sigma(T) \subset \mathbb{T}$):
$\mathrm{LMean}^2\mathrm{RG}(T)$ & $\mathrm{LMean}^2\mathrm{RG}(T^*)$ & $\mathrm{LRG}(T^{-1}) \Rightarrow \mathrm{Sim}(T)$.

Theorem B

Assume that Ω is a $C^{1+\alpha}$ Jordan domain, $\Gamma = \partial\Omega$, and $\sigma(T) \subset \Gamma$. If

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for $\{\gamma_s\}_{0<s<1}$ tending nicely to Γ from the outside, then T is similar to a normal.

Generalization of Naboko (1984)

Naboko (for $\sigma(T) \subset \mathbb{T}$): $\text{LMean}^2\text{RG}(T)$ & $\text{LMean}^2\text{RG}(T^{*-1}) \Rightarrow \text{Sim}(T)$.

Theorem C

Assume that Ω is a $C^{1+\alpha}$ Jordan domain, $\Gamma = \partial\Omega$, and $\sigma(T) \subset \Gamma$. If

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for $\{\gamma_s\}_{0<s<1}$ tending nicely to Γ from the outside and $\{\widetilde{\gamma_s}\}_{0<s<1}$ tending nicely to Γ from the inside, then T is similar to a normal.

Summary

Pseudoanalytic extension: If $f \in \mathbb{C}^{1+\alpha}(\Gamma)$, there is $F \in C^1(\mathbb{C})$ such that $F|\Gamma = f$ and $|\overline{\partial} F(z)| \leq C\|f\|_{C^{1+\alpha}} \operatorname{dist}(z,\Gamma)^\alpha$.

If $\|(T-\lambda)^{-1}\| \leq C \operatorname{dist}(\lambda,\Gamma)^{-1}$, Dynkin defines a $C^{1+\alpha}(\Gamma)$-calculus for T:

$$f(T) = \frac{1}{2\pi i}\int_{\partial D} F(\lambda)(\lambda - T)^{-1}\,d\lambda - \frac{1}{\pi}\iint_D \overline{\partial} F(\lambda)(\lambda - T)^{-1}\,dA(\lambda),$$

where F is a pseudoanalytic extension of f and $D \supset \Gamma$.

This calculus defines a map $\Phi : C^{1+\alpha}(\Gamma) \to B(H)$ which is linear, bounded, multiplicative, and such that $\Phi(z) = T$. It also satisfies the spectral mapping property: $\sigma(f(T)) = f(\sigma(T))$.

Pseudoanalytic extension and Dynkin's calculus

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Generalization of van Casteren (1980)

van Casteren (for $\sigma(T) \subset \mathbb{T}$): PowBd($T$) & LRG($T^{-1}$) $\Rightarrow$ Sim(T).

Theorem D (Dritschel-E.-Yakubovich arXiv:1510.08350)

Assume that Ω is a $C^{1+\alpha}$ Jordan domain, $\Gamma = \partial\Omega$, $\sigma(T) \subset \Gamma$, and $\eta : \Gamma \to \mathbb{T}$ a $C^{1+\alpha}$-diffeomorphism. Assume that $\|(T-\lambda)^{-1}\| \leq C \operatorname{dist}(\lambda, \Gamma)^{-1}$. Then $\eta(T)$ is defined by Dynkin's calculus. If $\eta(T)$ is power bounded, then T is similar to a normal.

Passing between Γ and $\mathbb{T}$

We take $\eta : \Gamma \to \mathbb{T}$ a $C^{1+\alpha}$-diffeomorphism and do a "good" pseudonalytic extension to a neighbourhood of Γ (we also denote it by η).

Key estimate:

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A generalization of a Theorem of B.Delyon-F.Delyon and Putinar-Sandberg

If Ω is a bounded convex domain and the of the numerical range of T is contained in $\overline{\Omega}$, then $\overline{\Omega}$ is a complete K-spectral set for T, for some $K \geq 1$.

Recall that the conclusion means that

$$\|f(T)\| \leq K\|f\|_{L^\infty(\overline{\Omega})},$$

for every matrix-valued rational function f with poles off $\overline{\Omega}$.

Theorem E

If Ω is $C^{1+\alpha}$ Jordan domain, $\Gamma = \partial\Omega$, $\sigma(T) \subset \overline{\Omega}$, and $R > 0$ is such that for each $\lambda \in \Gamma$ there is $\mu(\lambda) \in \mathbb{C} \setminus \overline{\Omega}$ such that $\operatorname{dist}(\mu(\lambda), \Gamma) = |\lambda - \mu(\lambda)| = R$ and $\|(T - \mu(\lambda))^{-1}\| \leq R^{-1}$. Then $\overline{\Omega}$ is a complete K-spectral set for T, for some $K \geq 1$.

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