Generation of algebras and separation of singularities

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Summary

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Generation of algebras in planar domains

- $\Omega \subset \mathbb{C}$ a domain
- $\mathfrak{A}$ a uniform algebra of analytic functions in Ω, $\mathfrak{A} = H^\infty(\Omega)$ or $\mathfrak{A} = A(\overline{\Omega}) = \mathcal{H}(\Omega) \cap C(\overline{\Omega})$
- $\Phi \subset \mathfrak{A}$ (tipically finite $\Phi = \{\varphi_1, \dots, \varphi_n\}$)

Denote by $\overline{\mathfrak{A}_\Phi}$ the smallest closed (or weak* closed) subalgebra of $\mathfrak{A}$ containing Φ.

Natural questions:

- When $\overline{\mathfrak{A}_\Phi} = \mathfrak{A}$?
- When $\overline{\mathfrak{A}_\Phi}$ has finite codimension in $\mathfrak{A}$?

Several papers study algebras of type $A(\overline{\Omega})$ and give sufficient conditions for $\overline{\mathfrak{A}_\Phi} = \mathfrak{A}$ (Wermer, Bishop, Blumenthal, Sibony-Wermer).
However, even in the simple case $\mathfrak{A} = A(\overline{\mathbb{D}})$, $\Phi = \{\varphi_1, \varphi_2\}$, a set of necessary and sufficient conditions for $\overline{\mathfrak{A}_\Phi} = \mathfrak{A}$ is not known.

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A different kind of subalgebra

Note that every $f \in \overline{\mathfrak{A}_\Phi}$ is a limit of polynomials in $\varphi_1, \ldots, \varphi_n$.

Assume that $\varphi_k : \Omega \to \overline{\mathbb{D}}$. We can define:

- $\mathcal{A}_\Phi$ the smallest subalgebra of $A(\overline{\Omega})$ containing all functions $g \circ \varphi_k$, $g \in A(\overline{\mathbb{D}})$
- $\mathcal{H}_\Phi$ the smallest subalgebra of $H^\infty(\Omega)$ containing all functions $g \circ \varphi_k$, $g \in H^\infty(\mathbb{D})$

These are not necessarily closed.

$f \in \mathcal{A}_\Phi$ is of the form

$$f(z) = \sum_{k=1}^{N} g_{1,k}(\varphi_1(z))g_{2,k}(\varphi_2(z)) \cdots g_{n,k}(\varphi_n(z)), \qquad g_{j,k} \in A(\overline{\mathbb{D}}).$$

Remark:

- If $\mathfrak{A} = A(\overline{\Omega})$, then $\mathcal{A}_\Phi \subset \overline{\mathfrak{A}_\Phi}$
- If $\mathfrak{A} = H^\infty(\overline{\Omega})$, then $\mathcal{H}_\Phi \subset \overline{\mathfrak{A}_\Phi}$

These algebras have applications to Operator Theory and to the study of uniform algebras in analytic curves.

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These algebras have applications to Operator Theory and to the study of uniform algebras in analytic curves.

- $\mathcal{V} \subset \mathbb{D}^n$ an analytic curve inside the polydisc
- Algebras $H^\infty(\mathcal{V})$ and $A(\overline{\mathcal{V}})$

Natural question: describe this algebras.

An example:

- $\Omega \subset \mathbb{C}$ a domain
- $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$
- Put $\mathcal{V} = \Phi(\Omega)$

The pullback $\Phi^* f = f \circ \Phi$.

- $\Phi^* A(\overline{\mathcal{V}})$ is a subalgebra of $A(\overline{\Omega})$
- $\Phi^* H^\infty(\overline{\mathcal{V}})$ is a subalgebra of $H^\infty(\Omega)$

Question: describe these subalgebras.

One application: extension results. Prove that every $f \in H^\infty(\mathcal{V})$ can be extended to an F in some algebra of functions in $\mathbb{D}^n$.

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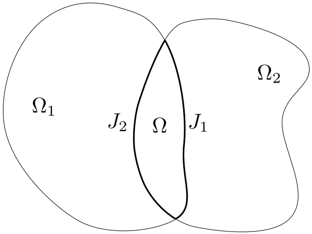
A simple example

- Ω_1, Ω_2 two Jordan domains
- $\Omega = \Omega_1 \cap \Omega_2$
- $\varphi_k : \Omega_k \to \mathbb{D}$, $k = 1, 2$, Riemann mappings



We want to write $f \in H^\infty(\Omega)$ as

$$f(z) = g_1(\varphi_1(z)) + g_2(\varphi_2(z)), \qquad g_1, g_2 \in H^\infty(\mathbb{D}).$$

Since φ_k are univalent, putting $g_k = h_k \circ \varphi_k$, this is equivalent to

$$f(z) = h_1(z) + h_2(z), \qquad h_k \in H^\infty(\Omega_k).$$

This decomposition is a separation of singularities: In some sense, f is singular in $J_1 \cup J_2$ and h_k is singular only in J_k.

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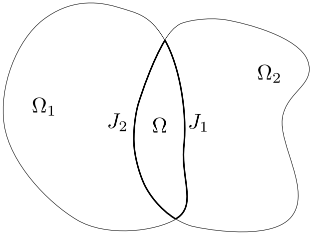

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Havin-Nersessian separation of singularities

First try:

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Put

$$h_k(z) = \int_{J_k} \frac{f(w)\,dw}{w-z}.$$

Then $f = h_1 + h_2$ and $h_k \in \mathcal{H}(\Omega_k)$.

However, $h_k \notin H^\infty(\Omega_k)$. h_k is not bounded near the endpoints of J_k, because it has singularities of logarithmic type there.

This simple procedure would have worked for H^p, $p < \infty$. But it does not work for H^∞.

We have to do something extra at the endpoints.

The idea of Havin-Nersessian: Put $\{z_1, z_2\} = J_1 \cap J_2$. Put $\Gamma_k = J_k \cap \mathbb{D}_\varepsilon(z_k)$. Let R_k be a rigid rotation around z_k such that $R_k(\Gamma_k)$ is outside Ω. Put

$$h_1(z) = \int_{J_1} \frac{f(w)\,dw}{w-z} + \int_{R_2(\Gamma_2)} \frac{f(R_2^{-1}(w))\,dw}{w-z} - \int_{R_1(\Gamma_1)} \frac{f(R_1^{-1}(w))\,dw}{w-z}.$$

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Non-univalent functions

We have proved: In the simple example $\Omega = \Omega_1 \cap \Omega_2$, $\Phi = \{\varphi_1, \varphi_2\}$ Riemann mappings, we have $\mathcal{H}_\Phi = H^\infty(\Omega)$. Even more is true: every $f \in H^\infty(\Omega)$ can be written as

$$f(z) = g_1(\varphi(z)) + g_2(\varphi(z)).$$

The same kind of arguments work when φ_k are univalent. But what about the case when φ_k are not univalent?

Two trivial remarks:

- If $\Phi = (\varphi_1, \ldots, \varphi_n)$ "glues" two points $z_1, z_2 \in \Omega$, i.e., $\Phi(z_1) = \Phi(z_2)$, then every $f \in \mathcal{H}_\Phi$ glues these two points
- If Φ' vanishes at some point $z_0 \in \Omega$, then $f'(z_0) = 0$ for every $f \in \mathcal{H}_\Phi$.

Even if Φ is injective and Φ' does not vanish, it is possible to show that in general one cannot hope to write every f as

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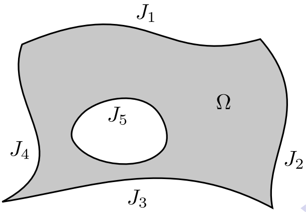
Summary

Admissible domains and maps

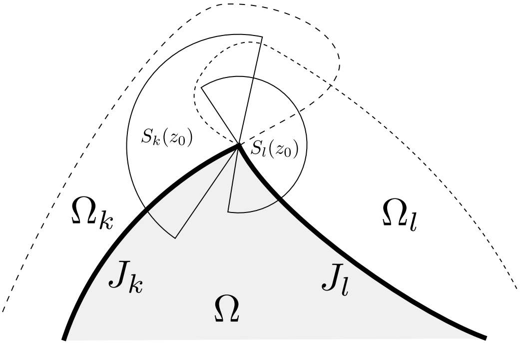
Definition

- $\Omega \subset \mathbb{C}$ a domain such that $\partial\Omega$ is a disjoint finite union of piecewise analytic Jordan curves. We assume that the interior angles of the "corners" of $\partial\Omega$ are between 0 and π.
- $\{J_k\}_{k=1}^n$ closed analytic arcs intersecting each other at most in two points and such that $\partial\Omega = \bigcup J_k$.
- $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ analytic in Ω, continuous up to the boundary plus some extra regularity conditions (see next slide).
- $|\varphi_k| = 1$ in J_k.
- φ_k' does not vanish in J_k.
- $\varphi_k(\zeta) \neq \varphi_k(z)$ if $\zeta \in J_k$, $z \in \overline{\Omega}$, and $z \neq \zeta$.



Regularity conditions

- For $k = 1, \ldots, n$, there is an open set $\Omega_k \supset \Omega$ such that the interior of J_k relative to $\partial\Omega$ is contained in Ω_k, $\varphi_k \in A(\overline{\Omega}_k)$ and φ_k' is of class Hölder α in Ω_k.
- If z_0 is an endpoint of J_k, then there exists an open circular sector $S_k(z_0)$ wih vertex on z_0 and such that $S_k(z_0) \subset \Omega_k$ and $J_k \cap \mathbb{D}_\varepsilon(z_0) \subset S_k(z_0) \cup \{z_0\}$, for some $\varepsilon > 0$. If z_0 is a common endpoint of J_k and J_l, we require $(S_k(z_0) \cap S_l(z_0)) \setminus \overline{\Omega} \neq \emptyset$.



Separation of singularities with the composition

Theorem

Let Ω and $\Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be admissible. Then there exist bounded linear operators $F_k : H^\infty(\Omega) \to H^\infty(\mathbb{D})$ such that the operator

$$f \mapsto f - \sum_{k=1}^{n} F_k(f) \circ \varphi_k$$

is compact in $H^\infty(\Omega)$ and its range is contained in $A(\overline{\Omega})$.
Moreover, F_k map $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$.

Techniques of the proof

- The integral operator

$$f \mapsto \int_{J_k} \left[\frac{1}{w-z} - \frac{\varphi_k'(w)}{\varphi_k(w) - \varphi_k(z)} \right] f(w)\, dw$$

is weakly singular. Hence compact.

- Replace the Cauchy integrals

$$\int_{J_k} \frac{1}{w-z} f(w)\, dw$$

by *modified* Cauchy integrals

$$\int_{J_k} \frac{\varphi_k'(w)}{\varphi_k(w) - z} f(w)\, dw,$$

which are analytic in $\mathbb{C} \setminus \varphi_k(J_k)$.

- Use the trick of Havin–Nersessian to get H^∞ functions when *cutting* f into a sum of Cauchy integrals in the arcs J_k.

Finite codimension

Theorem

If Ω and Φ are admissible, then $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$ are closed subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\overline{\Omega})$ respectively.

Proof.

Put $Gf = \sum_{k=1}^{n} F_k(f) \circ \varphi_k$. Then $G : H^\infty(\Omega) \to H^\infty(\Omega)$ and $G - I$ is compact. Hence, $GH^\infty(\Omega)$ is a closed subspace of finite codimension in $H^\infty(\Omega)$. Note that $GH^\infty(\Omega) \subset \mathcal{H}_\Phi$.

For $\mathcal{A}_\Phi$, use the restriction $G|A(\overline{\Omega})$. □

In fact, we also prove that $\mathcal{H}_\Phi$ is weak*-closed in $H^\infty(\Omega)$. The main idea in the proof of this is that many of our operators have a pre-adjoint operator.

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Equalities $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$

Theorem

If Ω and Φ are admissible, Φ is injective in $\overline{\Omega}$, and Φ' does not vanish in Ω, then $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$.

Recall: Φ being injective and Φ' not vanishing are necessary conditions for the equalities to hold.

The proof uses Banach algebra tools and the following classification of the one-codimensional closed unital subalgebras A_0 of a unital Banach algebra A (Gorin, 1969).

A_0 can have one of the following two forms:

- $A_0 = \ker(\psi_1 - \psi_2)$, where $\psi_1, \psi_2 \in \mathfrak{M}(A)$, $\psi_1 \neq \psi_2$. (Informally, A_0 are the functions which coincide at the points ψ_1 and ψ_2).
- $A_0 = \ker \eta$, where $\eta \neq 0$ is a continuous derivation at some $\psi \in \mathfrak{M}(A)$, i.e., $\eta \in A^*$ and

$$\eta(fg) = \eta(f)\psi(g) + \psi(f)\eta(g), \qquad \forall f, g \in A.$$

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Summary

1. Motivation: generation of algebras and algebras in analytic curves
2. Separation of singularities
3. Main results about generation of algebras
4. Main results about algebras in analytic curves
5. Consecuences for certain subalgebras of $H^\infty(\Omega)$

Algebras of functions in analytic curves

Recall: $\mathcal{V} = \Phi(\Omega)$ is an analytic curve in the polydisc $\mathbb{D}^n$. The pullback $\Phi^* f = f \circ \Phi$ takes function in $\mathcal{V}$ to functions in Ω.

Theorem

If Ω and Φ are admissible, then $\Phi^* H^\infty(\mathcal{V}) = \mathcal{H}_\Phi$ and $\Phi^* A(\overline{\mathcal{V}}) = \mathcal{A}_\Phi$.

The main tool of the proof is a characterization of the maximal ideal space and derivations of a *glued subalgebra*. This is a subalgebra B of an algebra A of the form

$$B = \{f \in A : \alpha_j(f) = \beta_j(f),\ j = 1, \dots, r\}, \qquad \alpha_j, \beta_j \in \mathfrak{M}(A),\ \alpha_j \neq \beta_j.$$

(Informally, B is the subalgreba of all functions in A which “glue” some determined points).

It turns out that $\mathfrak{M}(B)$ is obtained from $\mathfrak{M}(A)$ by gluing the points α_j and β_j. Also, the space of deriviations of B at a point $\psi_B \in \mathfrak{M}(B)$ is

$$\operatorname{Der}_{\psi_B}(B) \cong \bigoplus_{\psi \in (i^*)^{-1}(\psi_B)} \operatorname{Der}_\psi(A),$$

where $i^* : \mathfrak{M}(A) \to \mathfrak{M}(B)$ is the quotient map.

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The Agler algebra of $\mathbb{D}^n$

Is the algebra of analytic functions f in $\mathbb{D}^n$ such that the norm

$$\|f\|_{\mathcal{SA}(\mathbb{D}^n)} = \sup \|f(T_1, \dots, T_n)\|$$

is finite. The supremum is taken over all tuples $(T_1, \dots, T_n)$ of commuting contractions with $\sigma(T_j) \subset \mathbb{D}$.

- For every n, $\mathcal{SA}(\mathbb{D}^n) \subset H^\infty(\mathbb{D}^n)$
- For $n = 1$, there is equality and the norms are the same (von Neumann's inequality)
- For $n = 2$, there is equality and the norms are the same (Andô's theorem)
- For $n \geq 3$, the norms are different and it is believed that the inclusion is proper

Remark: every function which is a linear combination of functions which depend only on one or two of the variables belongs to $\mathcal{SA}(\mathbb{D}^n)$.

Extension to the Agler algebra

Theorem

If Ω and Φ are admissible, then every $f \in H^\infty(\mathcal{V})$ can be extended to an $F \in \mathcal{SA}(\mathbb{D}^n)$ with $\|F\|_{\mathcal{SA}(\mathbb{D}^n)} \leq C\|f\|_{H^\infty(\mathcal{V})}$. If f is continuous in $\overline{\mathcal{V}}$, then F can be taken to be continuous in $\overline{\mathbb{D}}^n$.

Idea of the proof: Take $f \in H^\infty(\mathcal{V})$. Then $\Phi^* f \in \mathcal{H}_\Phi$. We need to produce $F \in \mathcal{SA}(\mathbb{D}^n)$ such that $\Phi^* F = \Phi^* f$. If

$$(\Phi^* f)(z) = f_1(\varphi_1(z_1)) + \ldots + f_n(\varphi_n(z_n)), \tag{1}$$

just put $F(z_1, \ldots, z_n) = f_1(z_1) + \ldots + f_n(z_n)$.

The set of functions in $\mathcal{H}_\Phi$ that can be written as in (1) has finite codimension. We use Fredholm theory to extend our argument to all $\mathcal{H}_\Phi$.

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Previous extension results

Assume that there is an analytic variety $\widetilde{\mathcal{V}}$ in a neighbourhood of $\overline{\mathbb{D}}^n$ such that $\widetilde{\mathcal{V}} \cap \mathbb{D}^n = \mathcal{V}$.
Polyakov and Khenkin show that every $f \in H^\infty(\mathcal{V})$ can be extended to $F \in H^\infty(\mathbb{D}^n)$ with $\|F\|_{H^\infty(\mathbb{D}^n)} \leq C\|f\|_{H^\infty(\mathcal{V})}$.

- We do not assume the existence of $\widetilde{\mathcal{V}}$.
- We produce an extension $F \in \mathcal{SA}(\mathbb{D}^n)$.

Indeed, we show that there is a finite codimensional subspace of $H^\infty(\mathcal{V})$ such that every function in this subspace can be extended to an F of the form

$$F(z_1, \ldots, z_n) = F_1(z_1) + F_2(z_2) + \cdots + F_n(z_n), \qquad F_j \in H^\infty(\mathbb{D}),$$

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Summary

The algebra $H^\infty(\mathcal{K}_\Psi)$

- X a set, Ψ a collection of complex-valued functions in X
- $\sup\{|\psi(x)| : \psi \in \Psi\} < 1$, for all $x \in X$
- Ψ separates the points of X

Then Ψ is a *collection of test functions* on X.

A positive kernel $k : X \times X \to \mathcal{B}^*$ ($\mathcal{B}^*$ the dual of a C^*-algebra $\mathcal{B}$) is a function such that for every finite $F \subset X$ and $f : F \to \mathcal{B}$

$$\sum_{a,b \in F} k(a,b)(f(b)^* f(a)) \geq 0.$$

$\mathcal{K}_\Psi$ the collection of positive kernels k on X such that

$$(1 - \psi(x)\psi(y)^*)k(x,y)$$

is also positive for every $\psi \in \Psi$.
$H^\infty(\mathcal{K}_\Psi)$ the algebra of functions $f : X \to \mathbb{C}$ such that

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is positive for every $k \in \mathcal{K}_\Psi$ for some $C > 0$. The smallest such C gives $\|f\|_{H^\infty(\mathcal{K}_\Psi)}$.

Important applications in Operator Theory. Introduced by Dritschel and McCullough, 2007.

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The relation with $\mathcal{SA}(\mathbb{D}^n)$

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- $\mathcal{V} = \Phi(\Omega)$

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We always have the inclusions:

$$\mathcal{H}_\Phi \subset \Phi^* \mathcal{S}\mathcal{A}(\mathbb{D}^n) \subset \Phi^* H^\infty(\mathbb{D}^n) \subset \Phi^* H^\infty(\mathcal{V}) \subset H^\infty(\Omega).$$

In our setting, we have shown:

$$\mathcal{H}_\Phi = \Phi^* H^\infty(\mathcal{V})$$

and $\mathcal{H}_\Phi$ has finite codimension in $H^\infty(\Omega)$.
Moreover, if Φ is injective and Φ' does not vanish (i.e., if $\mathcal{V}$ is non-singular)

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Inclusions of several subalgebras of $H^\infty(\Omega)$

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