Separation of singularities, generation of algebras and complete $K$-spectral sets

Daniel Estévez

Universidad Autónoma de Madrid

Joint work with Michael Dritschel (Newcastle Univ.) and Dmitry Yakubovich (UAM)

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Summary

1. Test collections and complete $K$-spectral sets
2. Separation of singularities
3. Generation of algebras
4. Fitting everything together: idea of the proofs of the results about test functions
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1. Test collections and complete $K$-spectral sets
2. Separation of singularities
3. Generation of algebras
4. Fitting everything together: idea of the proofs of the results about test functions
If $T$ is a contraction on a Hilbert space $H$ (i.e., $\|T\| \leq 1$), then

$$\|p(T)\| \leq \max_{z \in \mathbb{D}} |p(z)|,$$

for every polynomial $p$.

In fact,

$$\|f(T)\|_{\mathcal{B}(H^s)} \leq \max_{z \in \mathbb{D}} \|f(z)\|,$$

for every rational function $f = [f_{jk}]_{j,k=1}^s$ with values on $s \times s$ matrices and no poles in $X$, and every $s \geq 1$.

Here, $f(T) = [f_{jk}(T)]_{j,k=1}^s$. 
Von Neumann’s inequality

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Complete $K$-spectral sets

**Definition**

$H$ a Hilbert space, $T \in \mathcal{B}(H)$ a bounded operator, $X \subset \hat{\mathbb{C}}$ a compact set. $X$ is a complete $K$-spectral set for $T$ if

$$\| f(T) \|_{\mathcal{B}(H^s)} \leq K \max_{z \in X} \| f(z) \|_{\mathcal{B}(\mathbb{C}^s)},$$

for every rational function $f = [f_{jk}]_{j,k=1}^s$ with values on $s \times s$ matrices and no poles in $X$, and every $s \geq 1$.

- $T$ is a contraction if and only if $\mathbb{D}$ is a complete 1-spectral set (von Neumann's inequality).
- $T$ is similar to a contraction ($T = SAS^{-1}$, $\|A\| \leq 1$) if and only if $\mathbb{D}$ is a complete $K$-spectral set for some $K$.
- $T$ is similar to an operator having a rational normal dilation to $\partial X$ if and only if $X$ is a complete $K$-spectral set for some $K$. This means that there is $\tilde{H} \supset H$ and $N \in \mathcal{B}(\tilde{H})$ normal with $\sigma(N) \subset \partial X$ such that

$$Sf(T)S^{-1} = P_H f(N)|H, \quad \forall f \text{ rational with no poles on } X.$$
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Some results about complete $K$-spectral sets

1. Let $\Omega_1, \ldots, \Omega_n \subset \hat{\mathbb{C}}$ be simply connected domains with analytic boundaries and such that their boundaries do not intersect. Then $\bigcap \Omega_j$ is complete $K$-spectral for $T$ if and only if $\overline{\Omega_j}$ is complete $K_j$-spectral for $T$. (Douglas, Paulsen, 1986).

2. Let $D_1, \ldots, D_n$ be discs in $\hat{\mathbb{C}}$. If $\overline{D_j}$ is (complete) 1-spectral for $T$, then $\bigcap D_j$ is complete $K$-spectral for $T$. (Badea, Beckermann, Crouzeix, 2009).

3. Let $X$ be a compact convex set. If the numerical range of $T$ 

$$W(T) = \{ \langle Tx, x \rangle : \|x\| = 1 \}$$

is contained in $X$, then $X$ is a complete $K$-spectral set for $T$. (Delyon, Delyon, 1999).

4. Let $B$ be a finite Blaschke product. If $\sigma(T) \subset \overline{D}$ and $\overline{D}$ is complete $K'$-spectral for $B(T)$, then $\overline{D}$ is complete $K$-spectral for $T$. (Mascioni, 1994).
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Some of our generalizations of these results

Theorem

Let $\Omega_1, \ldots, \Omega_s$ be Jordan domains with rectifiable and Ahlfors regular boundaries that intersect transversally. If $\overline{\Omega}_j$ is (complete) $K_j$-spectral for $T$, then $\bigcap \overline{\Omega}_j$ is (complete) $K$-spectral for $T$.

Theorem

Let $\Omega$ be a Jordan domain with $C^{1,\alpha}$ boundary. If $\overline{\Omega}$ and $\mathbb{C} \setminus \overline{\Omega}$ are $K$-spectral for $T$, then $\partial \Omega$ is complete $K'$-spectral for $T$. Hence, $T$ is similar to a normal operator with spectrum in $\partial \Omega$.

Theorem

Let $\Omega$ be a Jordan domain and $R > 0$ such that for each $\lambda \in \Omega$ there is $\mu \in \mathbb{C} \setminus \overline{\Omega}$ such that $B(\mu, R)$ is tangent to $\partial \Omega$ at $\lambda$. If $\| (T - \mu I)^{-1} \| \leq R^{-1}$, then $\overline{\Omega}$ is complete $K$-spectral for some $K > 0$.

If $\sigma(T) \subset \Gamma$ and $\| (T - z I)^{-1} \| \leq \text{dist}(z, \Gamma)^{-1}$, then $T$ is normal (Stampfli, 1969).
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If $\sigma(T) \subset \Gamma$ and $\|(T - zI)^{-1}\| \leq \text{dist}(z, \Gamma)^{-1}$, then $T$ is normal (Stampfli, 1969).
Our main problem:

\[ X \subset \hat{\mathbb{C}} \text{ some set. We look for a collection } \Phi \text{ of functions analytic in } X \text{ such that } \]

\[ \sigma(T) \subset X, \|\varphi(T)\| \leq 1, \forall \varphi \in \Phi \Rightarrow \overline{X} \text{ is complete } K\text{-spectral for } T, \quad (*) \]

or

\[ \sigma(T) \subset X, \overline{D} \text{ is complete } K'^{-}\text{spectral for } \varphi(T), \forall \varphi \in \Phi \Rightarrow \overline{X} \text{ is complete } K\text{-spectral for } T. \quad (**) \]

- Typically, \( X = \Omega \) an open domain, or \( X = \overline{\Omega} \).

**Definition**

- \( \Phi \) is a test collection in \( X \) if \( (*) \) holds.
- \( \Phi \) is a strong test collection in \( X \) if \( (**) \) holds.

There are different types of test collections depending on whether \( K \) can depend on \( T \) or not.
Test collections

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Let $\Omega_1, \ldots, \Omega_n \subset \hat{\mathbb{C}}$ be simply connected domains with analytic boundaries and such that their boundaries do not intersect. Let $\varphi_k : \overline{\Omega_k} \to \overline{\mathbb{D}}$ be Riemann conformal mappings. Then $\{\varphi_1, \ldots, \varphi_n\}$ is a strong test collection in $\bigcap \Omega_k$. \textit{(Douglas, Paulsen, 1986)}.

Let $D_1, \ldots, D_n$ be discs in $\hat{\mathbb{C}}$. Let $\varphi_k$ be a Möbius transformation taking $D_k$ onto $\mathbb{D}$. Then $\{\varphi_1, \ldots, \varphi_n\}$ is a test collection in $\bigcap D_k$. \textit{(Badea, Beckermann, Crouzeix, 2009)}.

Let $X$ be a compact convex set. Write $X = \bigcap H_\alpha$, with $H_\alpha$ closed half-planes. Let $\varphi_\alpha$ be a Möbius transformation taking $H_\alpha$ onto $\overline{\mathbb{D}}$. Then $\{\varphi_\alpha\}$ is a test collection in $X$. \textit{(Delyon, Delyon, 1999)}.

If $B$ is a finite Blaschke product, the set $\{B\}$ is a strong test collection in $\overline{\mathbb{D}}$. \textit{(Mascioni, 1994)}.
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Previous results restated in the language of test collections

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If $B$ is a finite Blaschke product, the set $\{B\}$ is a strong test collection in $\overline{\mathbb{D}}$. (Mascioni, 1994).
Definition

- $\Omega \subset \mathbb{C}$ a domain such that $\partial \Omega$ is a disjoint finite union of piecewise analytic Jordan curves. We assume that the interior angles of the “corners” of $\partial \Omega$ are between $0$ and $\pi$.

- $\{J_k\}_{k=1}^n$ closed analytic arcs intersecting each other at most in two points and such that $\partial \Omega = \bigcup J_k$.

- $\Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \rightarrow \mathbb{D}^n$ analytic in $\overline{\Omega}$ (can be weakened in many cases).

- $|\varphi_k| = 1$ in $J_k$.

- $\varphi'_k$ does not vanish in $J_k$.

- $\varphi_k(\zeta) \neq \varphi_k(z)$ if $\zeta \in J_k$, $z \in \overline{\Omega}$, and $z \neq \zeta$. 
A simple example of an admissible map

Example

$\Omega_1, \ldots, \Omega_n$ simply connected domains with analytic boundaries and such that their boundaries intersect transversally.

$\Omega = \bigcap \Omega_k$, $J_k = \partial \Omega \cap \partial \Omega_k$.

$\varphi_k : \overline{\Omega}_k \rightarrow \overline{D}$ Riemann conformal mappings.

But $\varphi_k$ need not be univalent in $\Omega$ in general.
A simple example of an admissible map

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But $\varphi_k$ need not be univalent in $\Omega$ in general.
Our results about admissible maps

Theorem A

Let $\Omega$ be a simply connected domain, and $\Phi : \overline{\Omega} \to \mathbb{D}^n$ admissible. Then $\Phi$ is a strong test collection in $\overline{\Omega}$ (with test collection constant depending on $T$). If moreover $\Phi$ is injective and $\Phi'$ does not vanish in $\Omega$, then the test collection constant does not depend on $T$.

Theorem B

Let $\Omega$ be a not necessarily simply connected domain. If $\Phi : \overline{\Omega} \to \mathbb{D}^n$ is admissible and injective and $\Phi'$ does not vanish in $\Omega$, then $\Phi$ is a strong test collection in $\Omega$. 
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3. Generation of algebras

4. Fitting everything together: idea of the proofs of the results about test functions
Let \( \Omega_1, \ldots, \Omega_n \) be simply connected domains with transversally intersecting boundaries. Put \( \Omega = \bigcap \Omega_k \). Then every \( f \in H^\infty(\Omega) \) can be written as

\[
f = f_1 + \cdots + f_n, \quad f_k \in H^\infty(\Omega_k).
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If \( \varphi_k : \Omega_k \to \mathbb{D} \) is a Riemann map, then every \( f \in H^\infty(\Omega) \) can be written as

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f = g_1 \circ \varphi_1 + \cdots + g_n \circ \varphi_n, \quad g_k \in H^\infty(\mathbb{D}).
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(Just put \( g_k = f_k \circ \varphi_k^{-1} \).)

Daniel Estévez (UAM)
Theorem (Havin–Nersessian–Ortega-Cerdà)

Let $\Omega_1, \ldots, \Omega_n$ be simply connected domains with transversally intersecting boundaries. Put $\Omega = \bigcap \Omega_k$. Then every $f \in H^\infty(\Omega)$ can be written as

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$$f = g_1 \circ \varphi_1 + \cdots + g_n \circ \varphi_n, \quad g_k \in H^\infty(\mathbb{D}).$$

(Just put $g_k = f_k \circ \varphi_k^{-1}$).
What can we do if $\varphi_k : \Omega \to \mathbb{D}$ are not univalent, but they still send $J_k$ bijectively onto some arc of $\mathbb{T}$?

**Theorem**

Let $\Omega$ and $\Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be admissible. Then there exist bounded linear operators $F_k : H^\infty(\Omega) \to H^\infty(\mathbb{D})$ such that the operator

$$f \mapsto f - \sum_{k=1}^{n} F_k(f) \circ \varphi_k$$

is compact in $H^\infty(\Omega)$ and its range is contained in $A(\overline{\Omega}) = \text{Hol}(\Omega) \cap C(\overline{\Omega})$. Moreover, $F_k$ map $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$. 

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The algebras $\mathcal{H}_\Phi$ and $A_\Phi$

$\Omega$ some domain, $\Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$.

$$\mathcal{H}_\Phi = \left\{ \sum_{j=1}^l f_{j,1}(\varphi_1(z)) f_{j,2}(\varphi_2(z)) \cdots f_{j,n}(\varphi_n(z)) : l \in \mathbb{N}, f_{j,k} \in H^\infty(\mathbb{D}) \right\}$$

$$A_\Phi = \left\{ \sum_{j=1}^l f_{j,1}(\varphi_1(z)) f_{j,2}(\varphi_2(z)) \cdots f_{j,n}(\varphi_n(z)) : l \in \mathbb{N}, f_{j,k} \in A(\overline{\mathbb{D}}) \right\}$$

These are the (non-closed) subalgebras of $H^\infty(\Omega)$ and $A(\overline{\Omega})$ generated by functions of the form $f \circ \varphi_k$, with $f \in H^\infty(\mathbb{D})$ or $f \in A(\overline{\mathbb{D}})$.

Questions:

- What geometric conditions on $\Phi$ guarantee that $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $A_\Phi = A(\overline{\Omega})$?
- What geometric conditions on $\Phi$ guarantee that $\mathcal{H}_\Phi$ and $A_\Phi$ are closed subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\overline{\Omega})$ (respectively)?
The algebras $\mathcal{H}_\Phi$ and $A_\Phi$

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If $\Omega$ and $\Phi$ are admissible, then $\mathcal{H}_\Phi$ and $A_\Phi$ are closed subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\overline{\Omega})$ respectively.

Proof.

Put $Gf = \sum_{k=1}^{n} F_k(f) \circ \varphi_k$. Then $G : H^\infty(\Omega) \to H^\infty(\Omega)$ and $G - I$ is compact. Hence, $GH^\infty(\Omega)$ is a closed subspace of finite codimension in $H^\infty(\Omega)$. Note that $GH^\infty(\Omega) \subset \mathcal{H}_\Phi$.

For $A_\Phi$, use the restriction $G|A(\overline{\Omega})$. 
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Equalities $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $A_\Phi = A(\overline{\Omega})$

**Theorem**

If $\Omega$ and $\Phi$ are admissible, $\Phi$ is injective in $\overline{\Omega}$, and $\Phi'$ does not vanish in $\Omega$, then $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $A_\Phi = A(\overline{\Omega})$.

Note: It is easy to see that $\Phi$ being injective and $\Phi'$ not vanishing are necessary conditions for the equalities to hold.

The proof uses Banach algebra tools and the classification of the one-codimensional closed unital subalgebras of a unital Banach algebra (Gorin, 1969).
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\( \mathcal{V} = \Phi(\Omega) \) is an analytic curve in the polydisc \( \mathbb{D}^n \). We consider the algebras \( H^\infty(\mathcal{V}) \) and \( A(\overline{\mathcal{V}}) \).

Put \( \Phi^* F = F \circ \Phi \).

**Theorem**

*If \( \Omega \) and \( \Phi \) are admissible, then \( \Phi^* H^\infty(\mathcal{V}) = \mathcal{H}_\Phi \) and \( \Phi^* A(\overline{\mathcal{V}}) = A_\Phi \).*

The Agler algebra of \( \mathbb{D}^n \):

\[
\|f\|_{SA(\mathbb{D}^n)} = \sup_{\|T_j\| \leq 1, \sigma(T_j) \subset \mathbb{D}} \|f(T_1, \ldots, T_n)\|.
\]

For every \( n \), \( SA(\mathbb{D}^n) \subset H^\infty(\mathbb{D}^n) \). For \( n = 1, 2 \), there is equality, but for \( n \geq 3 \), it is believed that the inclusion is proper.

**Theorem**

*If \( \Omega \) and \( \Phi \) are admissible, then every \( f \in H^\infty(\mathcal{V}) \) can be extended to an \( F \in SA(\mathbb{D}^n) \) with \( \|F\|_{SA(\mathbb{D}^n)} \leq C\|f\|_{H^\infty(\mathcal{V})} \).*
$\mathcal{V} = \Phi(\Omega)$ is an analytic curve in the polydisc $\mathbb{D}^n$. We consider the algebras $H^\infty(\mathcal{V})$ and $A(\overline{\mathcal{V}})$.
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1 Test collections and complete $K$-spectral sets

2 Separation of singularities

3 Generation of algebras

4 Fitting everything together: idea of the proofs of the results about test functions
Theorem B

If \( \Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n \) is admissible and injective and \( \Phi' \) does not vanish in \( \Omega \), then \( \Phi \) is a strong test collection in \( \Omega \).

Take \( T \) with \( \sigma(T) \subset \Omega \) and such that \( \mathbb{D} \) is complete \( K \)-spectral for \( \varphi_k(T) \), and \( f \) a \( s \times s \)-matrix–valued rational function with no poles in \( \overline{\Omega} \). We must show that

\[
\|f(T)\| \leq C \max_{z \in \Omega} \|f(z)\|.
\]

We do the case \( s = 1 \).

Put \( Gf = \sum F_k(f) \circ \varphi_k \). Since \( G - I \) is compact, there exist an operator \( R \) and an operator \( P \) with finite-dimensional range such that \( I = GR + P \). We can write

\[
f = \sum_{k=1}^{n} F_k(Rf) \circ \varphi_k + \sum_{j=1}^{r} \alpha_j(f)g_j,
\]

where \( \alpha_j \in (A(\overline{\Omega}))^* \) and \( g_j \in A(\overline{\Omega}) = A_\Phi \).

\[
\|g_j(T)\| = \left\| \sum_{t=1}^{l} f_{j,t,1}(\varphi_1(T)) \cdots f_{j,t,n}(\varphi_n(T)) \right\| \leq \sum_{t=1}^{l} K^n\|f_{j,t,1}\|_\infty \cdots \|f_{j,t,n}\|_\infty \leq C.
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Theorem B

If $\Phi : \bar{\Omega} \to \mathbb{D}^n$ is admissible and injective and $\Phi'$ does not vanish in $\Omega$, then $\Phi$ is a strong test collection in $\Omega$.

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\[ \| f(T) \| \leq \sum_{k=1}^{n} \| F_k(Rf)(\varphi_k(T)) \| + \sum_{j=1}^{r} |\alpha_j(f)| \| g_j(T) \| \leq C \| f \|_{\infty}. \]

The case \( s \geq 2 \) is the same. We have to use that an operator whose range is contained in a commutative \( C^* \)-algebra is automatically completely bounded. This means that the bounds that we have obtained before are uniform in \( s \).
The case when $\sigma(T) \cap \partial \Omega \neq \emptyset$

**Theorem A**

Let $\Omega$ be a simply connected domain, $\Phi : \overline{\Omega} \to \mathbb{D}^n$ admissible. Then $\Phi$ is a strong test collection in $\overline{\Omega}$ (with test collection constant depending on $T$). If moreover, $\Phi$ is injective and $\Phi'$ does not vanish in $\Omega$, then the test collection constant does not depend on $T$.

Here $\sigma(T)$ can intersect $\partial \Omega$. We cannot use the previous argument.

Idea: To use a *shrinking* of $\Omega$.

- $\{\psi_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ analytic and univalent functions on some open $U \supset \overline{\Omega}$.
- $\psi_0 \equiv z$.
- $\psi_\varepsilon(\overline{\Omega}) \subset \Omega$ for $\varepsilon > 0$.
- $\varepsilon \mapsto \psi_\varepsilon$ is continuous in the topology of uniform convergence on compact subsets of $U$.

To construct the shrinking we need that $\Omega$ is simply connected.

- Pass to operators $T_\varepsilon = \psi_\varepsilon(T)$.
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Thank you!