# Separation of singularities, generation of algebras and complete $K$-spectral sets

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# Summary

1. Test collections and complete $K$-spectral sets
2. Separation of singularities
3. Generation of algebras
4. Fitting everything together: idea of the proofs of the results about test functions

# Summary

# Von Neumann's inequality

If $T$ is a contraction on a Hilbert space $H$ (i.e., $\|T\| \leq 1$), then

$$\|p(T)\| \leq \max_{z \in \overline{\mathbb{D}}} |p(z)|,$$

for every polynomial $p$.

In fact,

$$\|f(T)\|_{\mathcal{B}(H^s)} \leq \max_{z \in \overline{\mathbb{D}}} \|f(z)\|,$$

for every for every rational function $f = [f_{jk}]_{j,k=1}^{s}$ with values on $s \times s$ matrices and no poles in $X$, and every $s \geq 1$.

Here, $f(T) = [f_{jk}(T)]_{j,k=1}^{s}$.

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# Complete $K$-spectral sets

## Definition

$H$ a Hilbert space, $T \in \mathcal{B}(H)$ a bounded operator, $X \subset \widehat{\mathbb{C}}$ a compact set. $X$ is a complete $K$-spectral set for $T$ if

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- $T$ is a contraction if and only if $\overline{\mathbb{D}}$ is a complete 1-spectral set (von Neumann's inequality).
- $T$ is similar to a contraction ($T = SAS^{-1}$, $\|A\| \leq 1$) if and only if $\overline{\mathbb{D}}$ is a complete $K$-spectral set for some $K$.
- $T$ is similar to an operator having a rational normal dilation to $\partial X$ if and only if $X$ is a complete $K$-spectral set for some $K$. This means that there is $\widetilde{H} \supset H$ and $N \in \mathcal{B}(\widetilde{H})$ normal with $\sigma(N) \subset \partial X$ such that

$$Sf(T)S^{-1} = P_H f(N)|H, \qquad \forall f \text{ rational with no poles on } X.$$

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# Some results about complete $K$-spectral sets

1. Let $\Omega_1, \ldots, \Omega_n \subset \widehat{\mathbb{C}}$ be simply connected domains with analytic boundaries and such that their boundaries do not intersect. Then $\overline{\bigcap \Omega_j}$ is complete $K$-spectral for $T$ if and only if $\overline{\Omega_j}$ is complete $K_j$-spectral for $T$. *(Douglas, Paulsen, 1986)*.
2. Let $D_1, \ldots, D_n$ be discs in $\widehat{\mathbb{C}}$. If $\overline{D_j}$ is (complete) 1-spectral for $T$, then $\overline{\bigcap D_j}$ is complete $K$-spectral for $T$. *(Badea, Beckermann, Crouzeix, 2009)*.
3. Let $X$ be a compact convex set. If the numerical range of T

$$W(T) = \{\langle Tx, x \rangle : \|x\| = 1\}$$

is contained in $X$, then $X$ is a complete $K$-spectral set for $T$. *(Delyon, Delyon, 1999)*.
4. Let $B$ be a finite Blaschke product. If $\sigma(T) \subset \overline{\mathbb{D}}$ and $\overline{\mathbb{D}}$ is complete $K'$-spectral for $B(T)$, then $\overline{\mathbb{D}}$ is complete $K$-spectral for $T$. *(Mascioni, 1994)*.

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# Some of our generalizations of these results

**Theorem**

*Let $\Omega_1, \ldots, \Omega_s$ be Jordan domains with rectifiable and Ahlfors regular boundaries that intersect transversally. If $\overline{\Omega}_j$ is (complete) $K_j$-spectral for $T$, then $\overline{\bigcap \Omega_j}$ is (complete) $K$-spectral for $T$.*

**Theorem**

*Let $\Omega$ be a Jordan domain with $C^{1,\alpha}$ boundary. If $\overline{\Omega}$ and $\mathbb{C} \setminus \overline{\Omega}$ are $K$-spectral for $T$, then $\partial\Omega$ is complete $K'$-spectral for $T$. Hence, $T$ is similar to a normal operator with spectrum in $\partial\Omega$.*

**Theorem**

*Let $\Omega$ be a Jordan domain and $R > 0$ such that for each $\lambda \in \Omega$ there is $\mu \in \mathbb{C} \setminus \overline{\Omega}$ such that $B(\mu, R)$ is tangent to $\partial\Omega$ at $\lambda$. If $\|(T - \mu I)^{-1}\| \leq R^{-1}$, then $\overline{\Omega}$ is complete $K$-spectral for some $K > 0$.*

If $\sigma(T) \subset \Gamma$ and $\|(T - zI)^{-1}\| \leq \operatorname{dist}(z, \Gamma)^{-1}$, then $T$ is normal *(Stampfli, 1969)*.

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# Test collections

**Our main problem:**

$X \subset \widehat{\mathbb{C}}$ some set. We look for a collection $\Phi$ of functions analytic in $X$ such that

$$\sigma(T) \subset X,\ \|\varphi(T)\| \leq 1, \forall \varphi \in \Phi \Rightarrow \overline{X} \text{ is complete } K\text{-spectral for } T, \qquad (*)$$

or

$$\sigma(T) \subset X, \overline{\mathbb{D}} \text{ is complete } K'\text{-spectral for } \varphi(T), \forall \varphi \in \Phi \Rightarrow \overline{X} \text{ is complete } K\text{-spectral for } T. \qquad (**)$$

- Tipically, $X = \Omega$ an open domain, or $X = \overline{\Omega}$.

**Definition**

- $\Phi$ is a test collection in $X$ if $(*)$ holds.
- $\Phi$ is a strong test collection in $X$ if $(**)$ holds.

There are different types of test collections depending on whether $K$ can depend on $T$ or not.

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# Previous results restated in the language of test collections

1. Let $\Omega_1, \ldots, \Omega_n \subset \widehat{\mathbb{C}}$ be simply connected domains with analytic boundaries and such that their boundaries do not intersect. Let $\varphi_k : \overline{\Omega_k} \to \overline{\mathbb{D}}$ be Riemann conformal mappings. Then $\{\varphi_1, \ldots, \varphi_n\}$ is a strong test collection in $\overline{\bigcap \Omega_k}$. *(Douglas, Paulsen, 1986)*.
2. Let $D_1, \ldots, D_n$ be discs in $\widehat{\mathbb{C}}$. Let $\varphi_k$ be a Möbius transformation taking $D_k$ onto $\mathbb{D}$. Then $\{\varphi_1, \ldots, \varphi_n\}$ is a test collection in $\overline{\bigcap D_k}$. *(Badea, Beckermann, Crouzeix, 2009)*.
3. Let $X$ be a compact convex set. Write $X = \bigcap H_\alpha$, with $H_\alpha$ closed half-planes. Let $\varphi_\alpha$ be a Möbius transformation taking $H_\alpha$ onto $\overline{\mathbb{D}}$. Then $\{\varphi_\alpha\}$ is a test collection in $X$. *(Delyon, Delyon, 1999)*.
4. If $B$ is a finite Blaschke product, the set $\{B\}$ is a strong test collection in $\overline{\mathbb{D}}$. *(Mascioni, 1994)*.

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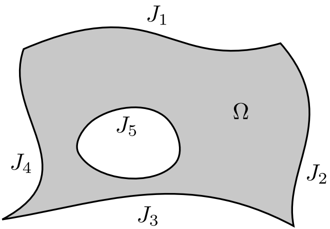
# Non-univalent test collections: Admissible domains and maps

## Definition

- $\Omega \subset \mathbb{C}$ a domain such that $\partial\Omega$ is a disjoint finite union of piecewise analytic Jordan curves. We assume that the interior angles of the "corners" of $\partial\Omega$ are between 0 and $\pi$.
- $\{J_k\}_{k=1}^n$ closed analytic arcs intersecting each other at most in two points and such that $\partial\Omega = \bigcup J_k$.
- $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ analytic in $\overline{\Omega}$ (can be weakened in many cases).
- $|\varphi_k| = 1$ in $J_k$.
- $\varphi_k'$ does not vanish in $J_k$.
- $\varphi_k(\zeta) \neq \varphi_k(z)$ if $\zeta \in J_k$, $z \in \overline{\Omega}$, and $z \neq \zeta$.



# A simple example of an admissible map

## Example

$\Omega_1, \ldots, \Omega_n$ simply connected domains with analytic boundaries and such that their boundaries intersect transversally.
$\Omega = \overline{\bigcap \Omega_k}$, $J_k = \partial\Omega \cap \partial\Omega_k$.
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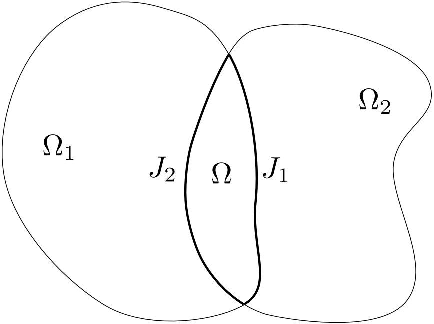


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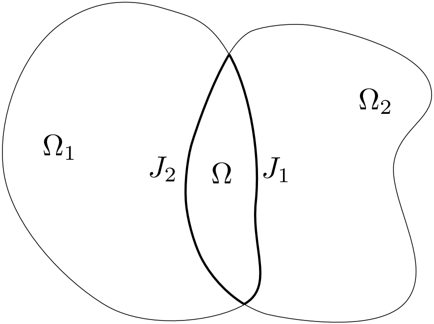


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# Our results about admissible maps

## Theorem A

Let $\Omega$ be a simply connected domain, and $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ admissible. Then $\Phi$ is a strong test collection in $\overline{\Omega}$ (with test collection constant depending on $T$). If moreover $\Phi$ is injective and $\Phi'$ does not vanish in $\Omega$, then the test collection constant does not depend on $T$.

## Theorem B

Let $\Omega$ be a not necessarily simply connected domain. If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and injective and $\Phi'$ does not vanish in $\Omega$, then $\Phi$ is a strong test collection in $\Omega$.

# Summary

# Havin–Nersesian–Ortega-Cerdà decomposition

**Theorem (Havin–Nersessian–Ortega-Cerdà)**

*Let $\Omega_1, \ldots, \Omega_n$ be simply connected domains with transversally intersecting boundaries. Put $\Omega = \bigcap \Omega_k$. Then every $f \in H^\infty(\Omega)$ can be written as*

$$f = f_1 + \cdots + f_n, \qquad f_k \in H^\infty(\Omega_k).$$

If $\varphi_k : \Omega_k \to \mathbb{D}$ is a Riemann map, then every $f \in H^\infty(\Omega)$ can be written as

$$f = g_1 \circ \varphi_1 + \cdots + g_n \circ \varphi_n, \qquad g_k \in H^\infty(\mathbb{D}).$$

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What can we do if $\varphi_k : \Omega \to \mathbb{D}$ are not univalent, but they still send $J_k$ bijectively onto some arc of $\mathbb{T}$?

**Theorem**

*Let $\Omega$ and $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be admissible. Then there exist bounded linear operators $F_k : H^\infty(\Omega) \to H^\infty(\mathbb{D})$ such that the operator*

$$f \mapsto f - \sum_{k=1}^{n} F_k(f) \circ \varphi_k$$

*is compact in $H^\infty(\Omega)$ and its range is contained in $A(\overline{\Omega}) = \mathrm{Hol}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Moreover, $F_k$ map $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$.*

# Separation of singularities with the composition

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*is compact in $H^\infty(\Omega)$ and its range is contained in $A(\overline{\Omega}) = \mathrm{Hol}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Moreover, $F_k$ map $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$.*

# Summary

1. Test collections and complete $K$-spectral sets
2. Separation of singularities
3. Generation of algebras
4. Fitting everything together: idea of the proofs of the results about test functions

# The algebras $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$

$\Omega$ some domain, $\Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$.

$$\mathcal{H}_\Phi = \left\{ \sum_{j=1}^{l} f_{j,1}(\varphi_1(z)) f_{j,2}(\varphi_2(z)) \cdots f_{j,n}(\varphi_n(z)) : \; l \in \mathbb{N}, \; f_{j,k} \in H^\infty(\mathbb{D}) \right\}$$

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These are the (non-closed) subalgebras of $H^\infty(\Omega)$ and $A(\overline{\Omega})$ generated by functions of the form $f \circ \varphi_k$, with $f \in H^\infty(\mathbb{D})$ or $f \in A(\overline{\mathbb{D}})$.

**Questions:**

- What geometric conditions on $\Phi$ guarantee that $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$?
- What geometric conditions on $\Phi$ guarantee that $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$ are closed subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\overline{\Omega})$ (respectively)?

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# Finite codimension

**Theorem**

*If $\Omega$ and $\Phi$ are admissible, then $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$ are closed subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\overline{\Omega})$ respectively.*

**Proof.**

Put $Gf = \sum_{k=1}^{n} F_k(f) \circ \varphi_k$. Then $G : H^\infty(\Omega) \to H^\infty(\Omega)$ and $G - I$ is compact. Hence, $GH^\infty(\Omega)$ is a closed subspace of finite codimension in $H^\infty(\Omega)$. Note that $GH^\infty(\Omega) \subset \mathcal{H}_\Phi$.

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# Equalities $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$

**Theorem**

*If $\Omega$ and $\Phi$ are admissible, $\Phi$ is injective in $\overline{\Omega}$, and $\Phi'$ does not vanish in $\Omega$, then $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$.*

Note: It is easy to see that $\Phi$ being injective and $\Phi'$ not vanishing are necessary conditions for the equalities to hold.

The proof uses Banach algebra tools and the classification of the one-codimensional closed unital subalgebras of a unital Banach algebra (Gorin, 1969).

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# Algebras of functions in analytic curves

$\mathcal{V} = \Phi(\Omega)$ is an analytic curve in the polydisc $\mathbb{D}^n$. We consider the algebras $H^\infty(\mathcal{V})$ and $A(\overline{\mathcal{V}})$.
Put $\Phi^* F = F \circ \Phi$.

**Theorem**

*If $\Omega$ and $\Phi$ are admissible, then $\Phi^* H^\infty(\mathcal{V}) = \mathcal{H}_\Phi$ and $\Phi^* A(\overline{\mathcal{V}}) = \mathcal{A}_\Phi$.*

The Agler algebra of $\mathbb{D}^n$:

$$\|f\|_{\mathcal{SA}(\mathbb{D}^n)} = \sup_{\substack{\|T_j\| \leq 1 \\ \sigma(T_j) \subset \mathbb{D}}} \|f(T_1, \dots, T_n)\|.$$

For every $n$, $\mathcal{SA}(\mathbb{D}^n) \subset H^\infty(\mathbb{D}^n)$. For $n = 1, 2$, there is equality, but for $n \geq 3$, it is believed that the inclusion is proper.

**Theorem**

*If $\Omega$ and $\Phi$ are admissible, then every $f \in H^\infty(\mathcal{V})$ can be extended to an $F \in \mathcal{SA}(\mathbb{D}^n)$ with $\|F\|_{\mathcal{SA}(\mathbb{D}^n)} \leq C\|f\|_{H^\infty(\mathcal{V})}$.*

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# Summary

## Theorem B

If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and injective and $\Phi'$ does not vanish in $\Omega$, then $\Phi$ is a strong test collection in $\Omega$.

Take $T$ with $\sigma(T) \subset \Omega$ and such that $\overline{\mathbb{D}}$ is complete $K$-spectral for $\varphi_k(T)$, and $f$ a $s \times s$-matrix–valued rational function with no poles in $\overline{\Omega}$. We must show that

$$\|f(T)\| \leq C \max_{z \in \overline{\Omega}} \|f(z)\|.$$

We do the case $s = 1$.
Put $Gf = \sum F_k(f) \circ \varphi_k$. Since $G - I$ is compact, there exist an operator $R$ and an operator $P$ with finite-dimensional range such that $I = GR + P$. We can write

$$f = \sum_{k=1}^{n} F_k(Rf) \circ \varphi_k + \sum_{j=1}^{r} \alpha_j(f) g_j,$$

where $\alpha_j \in (A(\overline{\Omega}))^*$ and $g_j \in A(\overline{\Omega}) = \mathcal{A}_\Phi$.

$$\|g_j(T)\| = \left\| \sum_{t=1}^{l} f_{j,t,1}(\varphi_1(T)) \cdots f_{j,t,n}(\varphi_n(T)) \right\| \leq \sum_{t=1}^{l} K^n \|f_{j,t,1}\|_\infty \cdots \|f_{j,t,n}\|_\infty \leq C.$$

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$$\|f(T)\| \leq \sum_{k=1}^{n} \|F_k(Rf)(\varphi_k(T))\| + \sum_{j=1}^{r} |\alpha_j(f)|\|g_j(T)\| \leq C\|f\|_\infty.$$

The case $s \geq 2$ is the same. We have to use that an operator whose range is contained in a commutative $C^*$-algebra is automatically completely bounded. This means that the bounds that we have obtained before are uniform in $s$.

## Theorem A

Let $\Omega$ be a simply connected domain, $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ admissible. Then $\Phi$ is a strong test collection in $\overline{\Omega}$ (with test collection constant depeding on $T$). If morever, $\Phi$ is injective and $\Phi'$ does not vanish in $\Omega$, then the test collection constant does not depend on $T$.

Here $\sigma(T)$ can intersect $\partial\Omega$. We cannot use the previous argument.

Idea: To use a *shrinking* of $\Omega$.

- $\{\psi_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$ analytic and univalent functions on some open $U \supset \overline{\Omega}$.
- $\psi_0 \equiv z$.
- $\psi_\varepsilon(\overline{\Omega}) \subset \Omega$ for $\varepsilon > 0$.
- $\varepsilon \mapsto \psi_\varepsilon$ is continuous in the topology of uniform convergence on compact subsets of $U$.

To construct the shrinking we need that $\Omega$ is simply connected.

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# The case when $\sigma(T) \cap \partial\Omega \neq \emptyset$

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Thank you!