Separating structures and operator vessels

Daniel Estévez

Universidad Autónoma de Madrid

Joint with Victor Vinnikov (Ben Gurion Univ. of the Negev) and Dmitry Yakubovich (UAM)

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Summary

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Definition (Commutative vessel)

H a Hilbert space, E a finite-dimensional Hilbert space.
$\Phi : H \to E$
Commuting operators $A_1, A_2 : H \to H$.
Selfadjoint operators $\sigma_1, \sigma_2, \gamma^{\text{in}}, \gamma^{\text{out}} : E \to E$.
The tuple $\mathcal{V} = (A_1, A_2; H, \Phi, E; \sigma_1, \sigma_2, \gamma^{\text{in}}, \gamma^{\text{out}})$ is a *commutative vessel* if:

$$\begin{aligned} \frac{1}{i}(A_k - A_k^*) &= \Phi^* \sigma_k \Phi, \\ \sigma_1 \Phi A_2^* - \sigma_2 \Phi A_1^* &= \gamma^{\text{in}} \Phi, \\ \gamma^{\text{out}} &= \gamma^{\text{in}} + i(\sigma_1 \Phi \Phi^* \sigma_2 - \sigma_2 \Phi \Phi^* \sigma_1), \\ \sigma_2 \Phi A_1 - \sigma_1 \Phi A_2 &= \gamma^{\text{out}} \Phi. \end{aligned}$$

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The discriminant curve

Discriminant polynomial:

$$\Delta(x_1, x_2) = \det(x_1\sigma_2 - x_2\sigma_1 + \gamma^{\text{in}}) = \det(x_1\sigma_2 - x_2\sigma_1 + \gamma^{\text{out}}).$$

Discriminant curve:

$$X = \{(x_1, x_2) \in \mathbb{C}^2 : \Delta(x_1, x_2) = 0\}.$$

Theorem (Generalized Cayley-Hamilton)

$$\Delta(A_1, A_2) = 0, \qquad \Delta(A_1^*, A_2^*) = 0.$$

In particular, $\sigma(A_1, A_2) \subset X$.

The classical Cayley-Hamilton theorem is obtained putting $A_1 = A$, $A_2 = iI$, $E = H$ y $\Phi = I$.

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Summary

Recall: $S \in \mathcal{B}(H)$ is subnormal if $S = N|H$, $N \in \mathcal{B}(K)$ normal.
S is pure subnormal if it has no non-trivial reducing subspace H_0 such that $S|H_0$ is normal.

The minimal normal extension: $N = \begin{bmatrix} S'^* & 0 \\ X & S \end{bmatrix}$.

S is subnormal of finite type if

$$C = S^*S - SS^*$$

has finite rank.
If S is pure subnormal of finite type

$$K = H_{0,-} \oplus M_- \oplus M_+ \oplus H_{0,+},$$

with $M_+ = CH$, $\dim M_- = \dim M_+ < \infty$, and

$$N = \begin{bmatrix} * & * & 0 & 0 \\ * & \Lambda_{-1} & 0 & 0 \\ 0 & T_0 & \Lambda_0 & * \\ 0 & 0 & * & * \end{bmatrix}.$$

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$$X = \{(z, w) \in \mathbb{C} : \det(T_0 T_0^* - (w - \Lambda_0)^*(z - \Lambda_0)) = 0\}.$$

$\sigma(N) \subset \{z \in \mathbb{C} : (z, \overline{z}) \in X\}$.

Equation rewritten in a form similar to vessels:

$$\det(x_1\sigma_2 - x_2\sigma_1 + \gamma) = 0,$$

where $z = x_1 + ix_2$, $w = x_1 - ix_2$, and

$$\sigma_1 = \begin{bmatrix} 0 & -iT_0^* \\ iT_0 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & T_0^* \\ T_0 & 0 \end{bmatrix}, \qquad \gamma = -\begin{bmatrix} T_0^* T_0 & T_0^* \Lambda_0 \\ T_0 \Lambda_{-1}^* & T_0 T_0^* \end{bmatrix}.$$

The mosaic function of D. Xia:

$$\mu(z) = P_{M_+}(N - SP_H)(N - z)^{-1}|M_+.$$

The mosaic function and the discriminant curve can be used to construct an analytic model for S.

The discriminant curve and mosaic function of D. Xia

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Summary

Definition (Separating structure)

A *separating structure* is:

1. A Hilbert space K.
2. A pair of commuting selfadjoint opartors $A_1, A_2 \in \mathcal{B}(K)$.
3. A decomposition
$$K = H_{0,-} \oplus M_- \oplus M_+ \oplus H_{0,+}$$
$\dim M_- = \dim M_+ < \infty$ and A_1, A_2 have the structure
$$A_j = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}, \qquad j = 1, 2.$$

Examples:

- N the minimal normal extension of a pure subnormal operator of finite type. $A_1 = \mathrm{Re}(\xi_1 N + \xi_2 N^*)$, $A_2 = \mathrm{Im}(\xi_1 N + \xi_2 N^*)$ form a separating structure ($\xi_j \in \mathbb{C}$).
- f rational with $|f| = 1$ in $\mathbb{T}$. $f^*(z) = \overline{f(\overline{z}^{-1})}$. $A_1 = M_{f+f^*}$, $A_2 = M_{(f-f^*)/i}$ in $L^2(\mathbb{T})$ form a separating structure.

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We define selfadjoint matrices $\sigma_1, \sigma_2, \gamma$ by

$$\sigma_j P_M = -i(P_{H_+} A_j - A_j P_{H_+}), \qquad \gamma P_M = i(A_1 P_{H_+} A_2 - A_2 P_{H_+} A_1).$$

Lemma

The selfadjoint matrices $\sigma_1, \sigma_2, \gamma \in \mathcal{B}(M)$ satisfy

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We consider the *discriminant curve*:

$$X = \{(x_1, x_2) \in \mathbb{C}^2 : \det(x_1\sigma_2 - x_2\sigma_1 + \gamma) = 0\}.$$

The joint spectrum of (A_1, A_2) is contained in the real part of the curve, $X \cap \mathbb{R}^2$.

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An algebraic curve is *separated* if its real part divides each component of the curve into two connected components, called *halves*.

In the case of separating structures, the algebraic curve is separated if certain weak assumptions hold.

Whenever the curve is separated, it should be possible to construct an analytic model for the operators (A_1, A_2) using Hardy H^2 spaces in the halves of the curve.

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The mosaic function

$N = A_1 + iA_2$ is normal.

Definition (Mosaic function)

$$\nu(z) = P_M(N - z)^{-1} P_{H_+}(N - z)|M \in \mathcal{B}(M), \qquad z \in \mathbb{C} \setminus \sigma(N).$$

$\nu(z)$ is projection-valued.

Theorem

The transform $V : K \to \mathrm{Hol}(\mathbb{C} \setminus \sigma(N))$,

$$(Vx)(z) = P_M(N - z)^{-1} x$$

"almost diagonalizes" N:

$$(VNx)(z) = zf(z) - [zf(z)]_{z=\infty}, \qquad f = Vx.$$

V *transforms* P_{H_+} *into multiplication by* ν:

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for $z, w \notin \sigma(N)$ *and* $(z, \overline{w}) \notin X$.

The mosaic $\nu(z)$, and matrices $\sigma_1, \sigma_2, \gamma$ determine the separating structure.

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The mosaic $\nu(z)$, and matrices $\sigma_1, \sigma_2, \gamma$ determine the separating structure.

Define the meromorphic function Q in X, whose values are parallel projections in M by

$$Q(x_1, x_2) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}(x_1 - ix_2, \varepsilon)} [\lambda(\sigma_1 + i\sigma_2) - (x_1 + ix_2)(\sigma_1 - i\sigma_2) - 2i\gamma]^{-1}(\sigma_1 + i\sigma_2)\, d\lambda.$$

Then

$$M = \sum_{\substack{(x_1, x_2) \in X \\ x_1 + ix_2 = z}} Q(x_1, x_2) M, \qquad \forall z \in \mathbb{C}.$$

Theorem (Restoration formula)

The curve X is separated. There exists a "half" X_+ of X such that

$$\nu(z) = \sum_{\substack{(x_1, x_2) \in X_+ \\ x_1 + ix_2 = z}} Q(x_1, x_2).$$

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Summary

Generalized compression

$K \supset H \supset G$ vector spaces. $A : K \to K$
The compression: $\widetilde{A} : H/G \to H/G$.
Hypotheses:

$$AG \cap H \subset G, \tag{C1}$$

$$AH \subset AG + H. \tag{C2}$$

Definition

We define the compression $\widetilde{A} : H/G \to H/G$ by:

1. Given $h \in H$, find $g \in G$ such that $h' \stackrel{\text{def}}{=} A(h-g) \in H$ (by (C2)).
2. $\widetilde{A}(h+G) \stackrel{\text{def}}{=} h' + G$.

This is well defined by (C1).

If $K = H_1 \oplus H_2 \oplus H_3$ and

$$A = \begin{bmatrix} * & 0 & 0 \\ * & A_0 & 0 \\ * & * & * \end{bmatrix},$$

the classical compression A_0 can be obtained by putting $G = H_3$, $H = H_2 \oplus H_3$.

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$A_1, A_2 : K \to K$. Two separating structures for A_1, A_2

$$\Sigma : \quad K = \overbrace{H_{0,-} \oplus M_-}^{H_-} \oplus \overbrace{M_+ \oplus H_{0,+}}^{H_+},$$

$$\widehat{\Sigma} : \quad K = \overbrace{\widehat{H}_{0,-} \oplus \widehat{M}_-}^{\widehat{H}_-} \oplus \overbrace{\widehat{M}_+ \oplus \widehat{H}_{0,+}}^{\widehat{H}_+},$$

such that

$$H_- \subset \widehat{H}_-, \qquad \widehat{H}_+ \subset H_+.$$

Lemma

The operators A_1, A_2 can be compressed to $H_+/\widehat{H}_+$ if and only if $P_{M_+}|\widehat{M}_+ : \widehat{M}_+ \to M_+$ is invertible.

Relation between separating structures and vessels

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Denote by $\widetilde{A}_1, \widetilde{A}_2$, the generalized compressions of A_1, A_2 to $H_+/\widehat{H}_+$.

Theorem

Given Σ, $\widehat{\Sigma}$, the following compressions are vessels:

$$\Sigma/\widehat{\Sigma} = \left(\widetilde{A}_1^*, \widetilde{A}_2^*; H_+/\widehat{H}_+, \widetilde{\Phi}, M; \sigma_1, \sigma_2, \gamma^{in} = \gamma - i(\sigma_1 \widetilde{\Phi}\widetilde{\Phi}^* \sigma_2 - \sigma_2 \widetilde{\Phi}\widetilde{\Phi}^* \sigma_1), \gamma^{out} = \gamma\right)$$

$$\Sigma \backslash \widehat{\Sigma} = \left(\widetilde{A}_1, \widetilde{A}_2; H_+/\widehat{H}_+, -\widetilde{\Phi}, M; -\sigma_1, -\sigma_2, \gamma^{in} = -\gamma, \gamma^{out} = -\gamma + i(\sigma_1 \widetilde{\Phi}\widetilde{\Phi}^* \sigma_2 - \sigma_2 \widetilde{\Phi}\widetilde{\Phi}^* \sigma_1)\right)$$

Note that the discriminant curves of $\Sigma/\widehat{\Sigma}$ and $\Sigma \setminus \widehat{\Sigma}$ are the same as that of Σ.

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Thank you!