THE HODGE THEOREM

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Abstract. In these notes we give a more or less self-contained proof of the Hodge theorem, using functional analysis techniques and Sobolev spaces of differential forms. The needed material, such as the definition of the inner product of forms, the Hodge $*$ operator and the definition of the Laplace-Beltrami operator are developed in detail. The reader is assumed to have certain basic background on differential geometry, but almost no previous knowledge of operator theory is needed. We also give applications of the theorem to cohomology.

1. Introduction

If M is a compact oriented Riemannian manifold of dimension n, the metric structure on M induces an inner product on each cotangent space T_p^*M. This inner product can be extended in a sensible way to the exterior algebra $\Lambda(T_p^*M)$, so that we are able to define an inner product on $\Omega(M)$ the space differential forms in M. Then one can also define a $*$ operator which takes k forms to $n-k$ forms and has some interesting properties in relation with this inner product.

Using the $*$ operator and the exterior derivative d, one can also define the Laplace-Beltrami operator Δ, which takes k forms to k forms. The Hodge theorem states that this operator Δ can be used to produce an orthogonal sum decomposition of $\Omega(M)$ (with respect to the inner product introduced above). This decomposition can be used to study when the non-homogeneus problem

$$\Delta\omega = \eta$$

can be solved. Here $\omega \in \Omega(M)$ is the unknown and $\eta \in \Omega(M)$ is some fixed form. One constructs Green's operator G, which, in some sense, can be used to solve this problem.

The forms satisfying the homogeneus problem

$$\Delta\omega = 0$$

are called harmonic. It turns out that these forms are related to the de Rham cohomology (for instance, harmonic forms are closed), so that the Hodge theorem has applications to cohomology. We will use it to give a proof of the Poincaré duality in a compact oriented manifold.

In Section 2 we give an exposition of all the material that will be needed to define the Laplace-Beltrami operator and state the Hodge theorem. The material treated here is fairly standard and is contained, for instance in [4] or [8] (although part of it appears only as exercises in both of these books).

In Section 3 we define the codifferential operator δ, which will play a very important role in the sequel, and the Laplace-Beltrami operator Δ. We prove a few basic properties of these operators and show their relation with the exterior derivative d. The exposition of this section follows that of the beginning of Chapter 6 in [8].

Section 4 is devoted to the statement of the Hodge theorem. Some comments regarding variations of the statement are also made. In order to prove the theorem, we will need to use the Sobolev spaces of differential forms. Their basic theory is developed in Section 5.

In Section 6 we turn out to prove the Hodge theorem, using functional analysis and operator theory tools. This Section and the preceding one are the densest of these notes, and both may be skipped on a first reading. The ideas in these two sections are taken from various sources. The main ideas are taken from [5] and [7], but the proof is original.

Green's operator is defined in Section 7, and some of its elementary properties are shown. Here we will need to use some results obtained in the proof of the Hodge theorem, but this Section can be read without reading Sections 5 and 6.

Date: January 11, 2013.

Finally, in Section 8 we give applications to cohomology. We show that there is a unique harmonic form in each cohomology class and we also give a proof of the Poincaré dualtiy. These last two sections also follow more or less the exposition in [8].

2. Preliminary material

In this section we will review all the material that we need to enounce the Hodge decomposition theorem. We will first define the exterior algebra of a vector space V, show how an inner product on V can be extended to its exterior algebra, define the Hodge $*$ operator and show some of its main properties.

Next, we will see how this constructions carry over to the differential forms in a compact oriented Riemannian manifold. This will allow us to give an inner product on the space of differential forms.

2.1. **The exterior algebra and the Hodge $*$ operator.**

2.1.1. *The exterior algebra.* Recall that if V is a vector space over a field K, we can define its k-th exterior power as the vector space

$$\Lambda^k(V) = \left\{ \sum_{j=1}^n \alpha_j v_{j_1} \wedge \cdots \wedge v_{j_k} :\ \alpha_j \in K,\ v_{j_r} \in V \right\},$$

where $v_1 \wedge \cdots \wedge v_k$ is a formal expression subject to the following algebraic properties:

$$\left(\sum_{j=1}^n \alpha_j v_j \right) \wedge w_2 \wedge \cdots \wedge w_k = \sum_{j=1}^n \alpha_j v_j \wedge w_2 \wedge \cdots \wedge w_k,$$
$$v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = -v_j \wedge \cdots \wedge v_1 \wedge \cdots \wedge v_k.$$

In other words, the expression $v_1 \wedge \cdots \wedge v_k$ is multilinear and alternate. By definition, the 0-th exterior power $\Lambda^0(V)$ is just the field K. Also, $\Lambda^1(V)$ is isomorphic to V. Observe that if V has finite dimension, then $\Lambda^k(V) = 0$ for $k > \dim V$ (recall that $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $v_1, \ldots, v_k$ are linearly dependent).

Next we define the exterior algebra of V as the direct sum of all the exterior powers of V:

$$\Lambda(V) = \sum_{k=0}^{\infty} \Lambda^k(V).$$

Observe that if V has finite dimension, the above sum is in fact finite and $\Lambda(V)$ has also finite dimension. The wedge product $\wedge$ turns $\Lambda(V)$ into a graded algebra.

2.1.2. *An inner product on* $\Lambda(V)$. Now we assume that the field K is either $\mathbb{R}$ or $\mathbb{C}$ and that V has an inner product $\langle \cdot, \cdot \rangle$. We want to extend this inner product to $\Lambda(V)$ (note that $\Lambda^1(V) = V$, so that V is a subspace of $\Lambda(V)$).

First we declare elements of different degree to be orthogonal. Hence, $\Lambda(V)$ will be the orthogonal sum of the exterior powers $\Lambda^k(V)$. Secondly, for elements $v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \in \Lambda^k(V)$, we define

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_j, w_l \rangle),$$

and extend this definition to $\Lambda^k(V)$ by linearity. It is easy to see that this defines an inner product on $\Lambda(V)$.

Observe that if V has finite dimension, then V is a Hilbert space and the above construction turns $\Lambda(V)$ into a Hilbert space, because it also has finite dimension. One can also check that in this case, the above construction is equivalent to declaring that if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V, then

$$\{e_{j_1} \wedge \ldots \wedge e_{j_k} :\ 1 \leq j_1 < j_2 < \cdots < j_k \leq n,\ k = 0, \ldots, n\} \tag{1}$$

is an orthonormal basis of $\Lambda(V)$.

2.1.3. *Oriented vector spaces.* We now recall the concept of an orientation on V. Assume that V is real and of finite dimension n. Since $\Lambda^n(V)$ is one-dimensional, $\Lambda^n(V)\setminus\{0\}$ has two connected components. An orientation on V is a choice of one of this two components, which we will call $\Lambda^n_+(V)$. Here we are tactily using the fact that given a finite dimensional vector space, there is a unique topology which turns it into a topological vector space. If the reader is not comfortable with this fact, he can just think of $\Lambda^n(V)$ as being $\mathbb{R}$.

Given a basis $v_1,\ldots,v_n$ of V, observe that if $w = v_1\wedge\cdots\wedge v_n$, then $w\neq 0$, so that w lies on one of the two components described above. We say that this basis is oriented if $w\in\Lambda^n_+(V)$. Note that the order of the elements of the basis is relevant here.

2.1.4. *The Hodge $*$ operator.* In the sequel we will assume that V is an oriented real Hilbert space of finite dimension n. Observe that

$$\dim\Lambda^k(V) = \dim\Lambda^{n-k}(V) = \binom{n}{k}, \qquad k = 0,\ldots,n. \tag{2}$$

Hence, it is possible to define a linear operator $*$ on $\Lambda(V)$ which takes $\Lambda^k(V)$ onto $\Lambda^{n-k}(V)$. We want this operator to have nice properties with respect to the Hilbert space structure on $\Lambda(V)$. This can be done in the following manner.

For any oriented orthonormal basis $e_1,\ldots,e_n$ of V, we put

$$*\,(e_1\wedge\cdots\wedge e_k) = e_{k+1}\wedge\cdots\wedge e_n, \qquad k = 0,\ldots,n. \tag{3}$$

Here we understand that the empty wedge product is the scalar $1\in\Lambda^0(V)$. It is easy to see that this is well defined, i.e., that if $f_1,\ldots,f_n$ is another oriented orthonormal basis such that $e_1\wedge\cdots\wedge e_k = f_1\wedge\cdots\wedge f_k$, then $e_{k+1}\wedge\cdots\wedge e_n = f_{k+1}\wedge\cdots\wedge f_n$.

Now fix an oriented orthonormal basis $\{e_1,\ldots,e_n\}$ of V. Note that (3) defines the action of $*$ on all the elements of the basis of $\Lambda(V)$ given in (1), so that it extends by linearity to an operator on $\Lambda(V)$. Indeed, let $e_{j_1}\wedge\ldots\wedge e_{j_k}$ be one of the elements given in (1). Assume $k<n$, since the case $k=n$ is trivial. We choose a permutation $\sigma\in\Sigma_n$ such that $\sigma(l) = j_l$ for $1\le l\le k$ and put $\varepsilon=\pm1$ such that $e_{\sigma(1)},\ldots,e_{\sigma(n-1)},\varepsilon e_{\sigma(n)}$ is an oriented basis. Then, by (3),

$$*(e_{j_1}\wedge\cdots\wedge e_{j_k}) = \varepsilon e_{\sigma(k+1)}\wedge\cdots\wedge e_{\sigma(n)}.$$

Therefore (3) defines an operator $*$ on $\Lambda(V)$ which takes $\Lambda^k(V)$ into $\Lambda^{n-k}(V)$ (we will immediately check that it takes $\Lambda^k(V)$ onto $\Lambda^{n-k}(V)$, as promised). Let us show some interesting properties of this operator. It is easy to see that

$$**|_{\Lambda^k(V)} = (-1)^{k(n-k)}I_{\Lambda^k(V)}, \tag{4}$$

i.e.,

$$**w = (-1)^{k(n-k)}w, \qquad w\in\Lambda^k(V).$$

This implies that $*$ is injective, so that indeed it takes $\Lambda^k(V)$ onto $\Lambda^{n-k}(V)$ by (2). This fact can also be seen directly from (3).

It is a cumbersome but easy computation to check that if $v,w\in\Lambda^k(V)$, then

$$\langle v,w\rangle = *(v\wedge *w). \tag{5}$$

This allows us to compute the scalar product by using solely the $*$ operator and the algebra structure of $\Lambda(V)$. Moreover, by using (5) with $v=w$ and taking (4) into account, we find that $*$ is an isometry:

$$\|*v\|^2 = *(*v\wedge **v) = (-1)^{k(n-k)}*(*v\wedge v) = *(v\wedge *v) = \|v\|^2.$$

This implies that $*$ is a unitary on $\Lambda(V)$ (this means that $*$ is isometric and onto).

2.2. **Compact oriented Riemmanian manifolds.** If M is a differentiable manifold, we define $\Lambda^k M$ as the k-th exterior power of T^*M, the cotangent bundle of M. This means that $\Lambda^k M$ is a vector bundle over M whose fibre at p is $\Lambda^k_p M = \Lambda^k(T^*_p M)$. We also put $\Lambda M = \sum \Lambda^k M$, the exterior algebra of T^*M. We denote by $\Omega^k(M)$ the space of smooth sections of $\Lambda^k M$. These are the so called differential forms of degree k. We also denote by $\Omega(M)$ the space of smooth sections of ΛM.

Throughout this notes, M will be a compact oriented Riemannian differentiable manifold of dimension n. Oriented means that we have chosen a non-vanishing $\omega \in \Lambda^n(M)$, which induces an orientation on each contangent space $T^*_p M$ by defining $\Lambda^n_+(T^*_p M)$ as the component in which ω_p lies. This also induces an orientation on each tangent space $T_p M$ by duality, but we will not need this fact.

Riemannian means that we have chosen an inner product $\langle \cdot, \cdot \rangle$ on each $T_p M$ in such a way that it "depends smoothly on p" (more formally, one fixes a metric g, which is a smooth section of $T^*M^{\otimes 2}$ the second tensor power of the cotangent bundle T^*M such that g_p gives a symmetric positive definite bilinear form on each $T_p M$).

2.2.1. *Canonical inner product on $T^*_p M$.* We want to see that the inner product on $T_p M$ given by the Riemannian structure of M induces an inner product on $T^*_p M$ in a canonical fashion. For this, we will use the following remark in functional analysis.

Remark. If H is a Hilbert space and H^* is its dual space, H^* is a Banach space with the norm

$$\|f\|_{H^*} = \sup_{\|x\|=1} |f(x)|.$$

The Riesz representation theorem furnishes a canonical map $\Phi : H^* \to H$ which is conjugate linear, isometric and onto. One can use this map to induce a canonical Hilbert space structure in H^* compatible with its Banach norm (meaning that $\langle f, f \rangle = \|f\|^2_{H^*}$). We just set

$$\langle f, g \rangle = \langle \Phi(g), \Phi(f) \rangle.$$

Since $T_p M$ is finite dimensional, it is a Hilbert space. By the above remark, $T^*_p M$ has also a Hilbert space structure, induced canonically by that of $T_p M$.

Alternatively, one can also do the following construction. Choose an orthonormal basis $\mathcal{B}$ of $T_p M$ and declare the corresponding dual basis of $T^*_p M$ to be orthonormal. This gives an inner product on $T^*_p M$ by linearity, and one can check that the inner product does not depend on the election of $\mathcal{B}$, so it is indeed canonical.

2.2.2. *The $*$ operator and the inner product on $\Omega(M)$.* We have seen that each cotangent space $T^*_p M$ is an oriented real Hilbert space of dimension n. Hence, we can apply the constructions of Section 2.1 on each fibre $\Lambda_p M = \Lambda(T^*_p M)$ of the bundle ΛM. Hence, the $*$ operator on each $\Lambda_p M$ gives rise to a bundle map on ΛM which we also call $*$. Therefore the operator $*$ also acts as a linear operator on $\Omega(M)$. This just means that for $\omega \in \Omega(M)$ we define $*\omega$ by $(*\omega)_p = *(\omega_p)$. Note that $*$ acting on $\Omega(M)$ is even $\mathcal{C}^\infty(M)$-linear.

The $*$ operator allows us to define integration of $\mathcal{C}^\infty$ functions on M. Observe that $\mathcal{C}^\infty(M) = \Omega^0(M)$, so that if $f \in \mathcal{C}^\infty(M)$, then $*f \in \Omega^n(M)$. Therefore, it makes sense to define

$$\int_M f = \int_M *f. \tag{6}$$

This definition is equivalent to the one involving the Riemannian volume form $d\mathrm{Vol}$. To see this, observe that by $\mathcal{C}^\infty(M)$-linearity it suffices to check that $*1 = d\mathrm{Vol}$, and this comes from (3).

Now that we have defined integration of $\mathcal{C}^\infty$ functions, we can define an inner product on $\Omega(M)$ using the inner product on each $\Lambda_p(M)$. We define

$$\langle \omega, \eta \rangle = \int_M \langle \omega, \eta \rangle, \qquad \omega, \eta \in \Omega(M). \tag{7}$$

Note that $\Omega(M)$ decomposes in orthogonal sum with respect to this inner product as

$$\Omega(M) = \bigoplus_{k=0}^{n} \Omega^k(M), \tag{8}$$

and that for forms of the same degree we can use (5) to obtain

$$\langle \omega, \eta \rangle = \int_M \omega \wedge *\eta, \qquad \omega, \eta \in \Omega^k(M). \tag{9}$$

Hence, once again, we can compute the inner product just by using the algebra structure of $\Omega(M)$ and the $*$ operator.

3. The codifferential and the Laplace-Beltrami operator

In this section we will define the Laplace-Beltrami operator Δ, which is the main protagonist of the Hodge theorem. We will also define the codifferential operator δ and prove some of the immediate properties of these two operators.

Recall that one of the most important operators on forms is the exterior derivative operator d. It is a linear operator on $\Omega(M)$ which takes $\Omega^k(M)$ into $\Omega^{k+1}(M)$. Now we will use the $*$ operator to define the so called codifferential operator δ. It will be a linear operator on $\Omega(M)$ taking $\Omega^k(M)$ into $\Omega^{k-1}(M)$ (we understand that $\Omega^{-1}(M) = 0$). It is defined by

$$\delta|_{\Omega^k(M)} = (-1)^{n(k+1)+1} *d*. \tag{10}$$

The definition extends by linearity to all $\Omega(M)$. Observe that using (4), it is easy to see that $\delta^2 = 0$.

The motivation for this definition is the following proposition.

Proposition 1. *If $\omega, \eta \in \Omega(M)$, then*

$$\langle d\omega, \eta \rangle = \langle \omega, \delta\eta \rangle.$$

Proof. By linearity and (8), we just need to check the case when $\omega \in \Omega^{k-1}(M)$ and $\eta \in \Omega^k(M)$. We first use (4) and (10) to compute

$$*\delta|_{\Omega^k(M)} = (-1)^{n(k+1)+1} **|_{\Omega^{n-k+1}(M)} d* = (-1)^{n(k+1)+1}(-1)^{(n-k+1)(k-1)} d* = (-1)^k d*.$$

Now, by using (9), the antiderivation property of d and Stoke's Theorem, we obtain

$$\langle d\omega, \eta \rangle = \int_M d\omega \wedge *\eta = (-1)^k \int_M \omega \wedge d*\eta + \int_M d(\omega \wedge *\eta) = \int_M \omega \wedge *\delta\eta = \langle \omega, \delta\eta \rangle. \qquad \square$$

In view of this proposition, it is usually said that δ is the adjoint of d. In this notes, we will reserve the term "adjoint" for the adjoint of a bounded operator defined on a Hilbert space (d is unbounded and it is defined on $\Omega(M)$, which is not complete). Hence, we will say that δ is the formal adjoint of d.

Now we are able to define the Laplace-Beltrami operator Δ and prove a few basic facts about it. Δ will be an operator on $\Omega(M)$ taking $\Omega^k(M)$ into itself. It is defined by

$$\Delta = d\delta + \delta d. \tag{11}$$

The next propositions gather some of its main properties.

Proposition 2. *The Laplace-Beltrami operator Δ is symmetric (i.e., formally self-adjoint). This means that*

$$\langle \Delta\omega, \eta \rangle = \langle \omega, \Delta\eta \rangle, \qquad \omega, \eta \in \Omega(M).$$

Proof. Using Proposition 1, we have

$$\langle \Delta\omega, \eta \rangle = \langle d\delta\omega, \eta \rangle + \langle \delta d\omega, \eta \rangle = \langle \omega, d\delta\eta \rangle + \langle \omega, \delta d\eta \rangle = \langle \omega, \Delta\eta \rangle. \qquad \square$$

A form $\omega \in \Omega(M)$ is said to be harmonic if $\Delta\omega = 0$, i.e., if $\omega \in \ker \Delta$. Harmonic forms will play an important role in the sequel. The next proposition relates the space of harmonic forms with the kernels of d and δ and will give a link between harmonic forms and cohomology (note that it proves that harmonic forms are closed).

Proposition 3. $\ker \Delta = \ker d \cap \ker \delta$, *i.e.,* $\Delta\omega = 0$ *if and only if* $d\omega = \delta\omega = 0$.

Proof. Assume that $\omega \in \ker \Delta$. Then

$$0 = \langle \Delta\omega, \omega \rangle = \langle d\delta\omega, \omega \rangle + \langle \delta d\omega, \omega \rangle = \|\delta\omega\|^2 + \|d\omega\|^2,$$

so that $\omega \in \ker d \cap \ker \delta$. This proves $\ker \Delta \subset \ker d \cap \ker \delta$. The reverse inclusion is trivial. $\square$

4. The Hodge Theorem

We are now ready to enounce the Hodge theorem. The proof will be delayed until section 6, where we will use tools from functional analysis to attack the problem and prove further properties of the Laplace operator.

Theorem 4 (Hodge Decomposition Theorem)**.** *The space* $\Omega(M)$ *decomposes in orthogonal sum as*

$$\Omega(M) = \Delta(\Omega(M)) \oplus \ker \Delta. \tag{12}$$

Moreover, $\ker \Delta$ *is finite-dimensional.*

4.1. **Alternative decompositions.** From this decomposition, a few others can be easily obtained by using the properties of d and δ. Let us comment the more usual.

First observe that $d^2 = 0$ implies that $d(\Omega(M)) \perp \delta(\Omega(M))$. Indeed,

$$\langle d\omega, \delta\eta \rangle = \langle d^2\omega, \eta \rangle = 0, \omega, \eta \in \Omega(M).$$

Also, $d(\Omega(M)) \perp \ker \Delta$, because if $\Delta\omega = 0$, then $\delta\omega = 0$ and

$$\langle d\eta, \omega \rangle = \langle \eta, \delta\omega \rangle = 0, \eta \in \Omega(M).$$

Similarly, $\delta(\Omega(M)) \perp \ker \Delta$.

This implies

$$\Omega(M) = d\delta(\Omega(M)) \oplus \delta d(\Omega(M)) \oplus \ker \Delta. \tag{13}$$

Here, the inclusion $\Delta(\Omega(M)) \subset d\delta(\Omega(M)) \oplus \delta d(\Omega(M))$ is trivial and the reverse inclusion comes from the fact that $d\delta(\Omega(M)) \oplus \delta d(\Omega(M)) \perp \ker \Delta$. Likewise, one also obtains

$$\Omega(M) = d(\Omega(M)) \oplus \delta(\Omega(M)) \oplus \ker \Delta. \tag{14}$$

It is also important to mention that using (8) and the fact that d takes $\Omega^k(M)$ into $\Omega^{k+1}(M)$ and δ takes $\Omega^k(M)$ into $\Omega^{k-1}(M)$, we also obtain orthogonal decompositions for each $\Omega^k(M)$. For instance, (12) yields

$$\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \ker(\Delta|_{\Omega^k(M)}),$$

and from (14) we get

$$\Omega^k(M) = d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus \ker(\Delta|_{\Omega^k(M)}).$$

4.2. **A completely wrong idea of the proof.** Here we will give an idea of the proof which is completely wrong, but which we cannot refrain from showing because it morally contains the essence of the proof that will be given in Section 6 and motivates the forthcoming exposition.

Assume that $\Omega(M)$ is a Hilbert space and that Δ is a bounded operator (of course this is the part that is completely wrong). Then Proposition 2 shows that Δ is self-adjoint. Hence,

$$\Omega(M) = \overline{\Delta(\Omega(M))} \oplus \ker \Delta^* = \overline{\Delta(\Omega(M))} \oplus \ker \Delta.$$

If we can show that Δ has closed range, then we have (12). Then we would have to prove somehow that $\ker \Delta$ has finite dimension.

As we have said, the problem with this argument is that $\Omega(M)$ is not complete and Δ is not bounded. The first problem can be solved if we can extend Δ to some Hilbert space containing $\Omega(M)$. The second obstruction can be remedied by changing the norm on the domain of Δ to make it bounded (in fact we will do so in a way which makes the boundedness of Δ trivial). However, in doing this, we are losing the self-adjointness of Δ (its domain and codomain will be different spaces).

We will also need to prove that the range of Δ is closed and its kernel has finite dimension. The first property will come (as is usual in functional analysis) from some lower bound on Δ and the second one will come from some compactness properties.

In our way to extend Δ to a bounded operator on a Hilbert space, we will need to introduce the Sobolev spaces of differential forms, which formalize the notion of differentiation of non-smooth forms in M.

5. Sobolev spaces of differential forms

In this section we will give the definition of the Sobolev spaces of differential forms, which are Hilbert spaces extending $\Omega(M)$ and on which it makes sense to define the operators d and δ. First we will need to introduce the Hilbert space of square-integrable forms in M, and for that, it is in order to make a few remarks about integration of functions on M.

5.1. **The space $L^2(\Lambda M)$.** It is an important observation that the definition of the integral of an $\omega \in \Omega^n(M)$ makes sense for a wider class of sections of the bundle $\Lambda^n M$ (recall that $\Omega^n(M)$ is the space of $\mathcal{C}^\infty$ sections of $\Lambda^n M$). Hence, using (6) we can define the integral of a scalar valued function f on M first for non-negative measurable functions (meaning that their coordinate expressions are Lebesgue measurable in each local chart) and then for measurable functions f such that the integral of $|f|$ is finite.

This allows us to define the space $L^2(\Lambda M)$ of L^2-forms on M as the spaces of measurable sections ω of the bundle ΛM such that

$$\int_M \|\omega\|^2 < \infty.$$

By measurable sections we mean, as above, that the coordinate expressions are Lebesgue measurable in each local chart. Then

$$\langle \omega, \eta \rangle_{L^2(\Lambda M)} = \int_M \langle \omega, \eta \rangle, \qquad \omega, \eta \in L^2(\Lambda M)$$

makes sense and defines an inner product which turns $L^2(\Lambda M)$ into a Hilbert space. $\Omega(M)$ is a dense subspace of $L^2(\Lambda M)$, so $L^2(\Lambda M)$ can be regarded as the completion of $\Omega(M)$ with respect to this inner product.

5.2. **The Sobolev spaces $W^{k,2}(\Lambda M)$.** Here we define the Sobolev spaces of differential forms using weak differentials and codifferentials. See [3, Section 3.2] for a brief introduction to this topic and [7] for a more in-depth treatment, including a proof of a Hodge decomposition theorem for $L^p(\Lambda M)$.

If $\omega \in L^2(\Lambda M)$, we say that $\eta \in L^2(\Lambda M)$ is the weak differential of ω if

$$\langle \omega, \delta\alpha \rangle_{L^2} = \langle \eta, \alpha \rangle_{L^2}, \qquad \forall \alpha \in \Omega(M).$$

Since $\Omega(M)$ is dense in $L^2(\Lambda M)$, the weak differential is unique (if it exists) and we write $\eta = d\omega$. Similarly, we say that $\eta \in L^2(\Lambda M)$ is the weak codifferential of ω and write $\eta = \delta\omega$ if

$$\langle \omega, d\alpha \rangle_{L^2} = \langle \eta, \alpha \rangle_{L^2}, \qquad \forall \alpha \in \Omega(M).$$

Proposition 1 implies that if $\omega \in \Omega(M)$, then $d\omega$ is its weak differential and $\delta\omega$ its weak codifferential (here d and δ are the operators considered in the preceding section), so this notation is justified.

Now we define $W^{1,2}(\Lambda M)$ the Sobolev space of forms with one derivative on M as the space of $\omega \in L^2(\Lambda M)$ such that ω has weak differential and codifferential and $d\omega, \delta\omega \in L^2(\Lambda M)$. We endow $W^{1,2}(\Lambda M)$ with the norm

$$\|\omega\|^2_{W^{1,2}(\Lambda M)} = \|\omega\|^2_{L^2(\Lambda M)} + \|d\omega\|^2_{L^2(\Lambda M)} + \|\delta\omega\|^2_{L^2(\Lambda M)}.$$

With this norm, $W^{1,2}(\Lambda M)$ is a Hilbert space and $\Omega(M)$ is dense in $W^{1,2}(\Lambda M)$.

We also define for integer $k \geq 2$ the Sobolev space of forms with k derivatives on M

$$W^{k,2}(\Lambda M) = \{\omega \in W^{1,2}(\Lambda M) : d\omega, \delta\omega \in W^{k-1,2}(\Lambda M)\},$$

together with the norm

$$\|\omega\|^2_{W^{k,2}(\Lambda M)} = \|\omega\|^2_{L^2(\Lambda M)} + \|d\omega\|^2_{W^{k-1,2}(\Lambda M)} + \|\delta\omega\|^2_{W^{k-1,2}(\Lambda M)}.$$

It is also a Hilbert space and $\Omega(M)$ is dense in $W^{k,2}(\Lambda M)$ for all k.

The following analogue of Proposition 1 is easy to obtain.

Proposition 5. *If $\omega, \eta \in W^{1,2}(\Lambda M)$, then*

$$\langle d\omega, \eta \rangle_{L^2} = \langle \omega, \delta\eta \rangle_{L^2}.$$

Proof. If $\eta \in \Omega(M)$, this is just the definition of $d\omega$. If $\eta \in W^{1,2}(\Lambda M)$, take $\varepsilon > 0$ and choose $\tilde{\eta} \in \Omega(M)$ with $\|\eta - \tilde{\eta}\|_{W^{1,2}} < \varepsilon$. Now

$$|\langle d\omega, \eta\rangle_{L^2} - \langle \omega, \delta\eta\rangle_{L^2}| \leq |\langle d\omega, \eta - \tilde{\eta}\rangle_{L^2}| + |\langle \omega, \delta(\eta - \tilde{\eta})\rangle_{L^2}| < \|d\omega\|_{L^2}\varepsilon + \|\omega\|_{L^2}\varepsilon.$$

Let $\varepsilon \to 0$. □

The definition of $W^{k,2}(\Lambda M)$ using weak differentials and codifferentials is different from the usual definition of Sobolev spaces of functions on $\mathbb{R}^n$, which uses all the partial derivatives in the definition of the norm (see [2, Chapter 5] for a basic introduction to Sobolev spaces in $\mathbb{R}^n$ and their applications to PDEs). A definition of $W^{k,2}(\Lambda M)$ which is more similar to that of $W^{k,2}(\mathbb{R}^n)$ can be made either by using covariant derivatives (see [1]) or by requiring that the local expressions of a form are in $W^{k,2}(\mathbb{R}^n)$ for each (sufficiently nice) local chart.

This alternative definitions may seem stronger at first sight, since they involve all the partial derivatives of the local expressions of ω and not only those which appear in $d\omega$ and $\delta\omega$. However, it turns out that all the definitions are equivalent. The key point in establishing this result is the proof of the so called Gaffney inequality. See [7] for a proof of this fact. This equivalence allows us to carry over to $W^{k,2}(\Lambda M)$ many of the well known properties of the Sobolev spaces in $\mathbb{R}^n$. In particular, we have the two following theorems, which we state without proof and will use in the sequel. The proof of these theorems for $\mathbb{R}^n$ can be found, for instance, in [2].

The first theorem is a compactness result which will be of key importance in the proof of the Hodge theorem.

Theorem 6 (Rellich-Kondrachov)**.** *The inclusion $W^{1,2}(\Lambda M) \hookrightarrow L^2(\Lambda M)$ is a compact operator. This means that if $\{\omega_n\}_{n=1}^{\infty} \subset W^{1,2}(\Lambda M)$ is a sequence such that $\|\omega_n\|_{W^{1,2}(\Lambda M)}$ is bounded, then there is a subsequence $\{\omega_{n_k}\}$ which converges in $L^2(\Lambda M)$.*

Using Banach-Alaoglu's theorem (see [6, Theorem 3.15]), we can get a subsequence with even better convergence properties. Recall that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in a Hilbert space, we say that x_n converges weakly to x if for every y we have $\langle x_n, y\rangle \to \langle x, y\rangle$. We also say that x_n converges strongly to x if $\|x_n - x\| \to 0$ (this is the type of convergence that we always consider unless stated otherwise).

Corollary 7. *If $\{\omega_n\}_{n=1}^{\infty} \subset W^{1,2}(\Lambda M)$ is a sequence such that $\|\omega_n\|_{W^{1,2}(\Lambda M)}$ is bounded, then there is a subsequence $\{\omega_{n_k}\}$ which converges weakly to some ω_0 in $W^{1,2}(\Lambda M)$ and converges strongly to ω_0 in $L^2(\Lambda M)$.*

Proof. By Banach-Alaoglu's theorem, there is a subsequence ω_{n_k} which converges weakly to some ω_0 in $W^{1,2}(\Lambda M)$. It is well known that compact operators map weakly convergent sequences into strongly convergent sequences. Hence, ω_{n_k} also converges strongly to ω_0 in $L^2(\Lambda M)$. □

The second result relates weak differentiability with the existence of classical derivatives. We will use it to find out which $\omega \in W^{k,2}(\Lambda M)$ turn out to be smooth (i.e., in $\Omega(M)$).

Theorem 8 (Sobolev embedding theorem)**.** *If $m < k - n/2$, then every $\omega \in W^{k,2}(\Lambda M)$ coincides a.e. with a $\mathcal{C}^m$ section of ΛM.*

Corollary 9. *We have*

$$\bigcap_{k=1}^{\infty} W^{k,2}(\Lambda M) = \Omega(M).$$

Proof. The preceding Theorem proves that if $\omega \in W^{k,2}(\Lambda M)$ for all $k \geq 1$, then ω coincides a.e. with a $\mathcal{C}^m$ section of ΛM for all $m \geq 0$, so that $\omega \in \Omega(M)$. The reverse inclusion is trivial. □

6. Proof of the Hodge theorem

We have now all the material we need to start the proof of Theorem 4. First observe that d and δ define bounded linear operators from $W^{1,2}(\Lambda M)$ to $L^2(\Lambda M)$ and from $W^{k,2}(\Lambda M)$ to $W^{k-1,2}(\Lambda M)$, $k \geq 2$ which extend the operators that were defined on $\Omega(M)$ in Section 3. Also, using (11) we can define a bounded Laplace operator $\Delta : W^{2,2}(\Lambda M) \to L^2(\Lambda M)$ which extends the one considered previously.

We denote by $\Delta^* : L^2(\Lambda M) \to W^{2,2}(\Lambda M)$ the adjoint operator of Δ. This means that Δ^* is the bounded operator defined by

$$\langle \Delta^* \omega, \eta \rangle_{W^{2,2}(\Lambda M)} = \langle \omega, \Delta \eta \rangle_{L^2(\Lambda M)}, \qquad \omega \in L^2(\Lambda M),\ \eta \in W^{2,2}(\Lambda M).$$

Motivated by Proposition 3 we define the space

$$\mathcal{H} = \{\omega \in W^{1,2}(\Lambda M) : d\omega = \delta\omega = 0\}. \tag{15}$$

Since an $\omega \in \mathcal{H}$ may be non-smooth (i.e. $\omega \notin \Omega(M)$), a priori, it may be possible that this space $\mathcal{H}$ is strictly larger than the space of harmonic (smooth) forms considered in Section 3. However, we will presently see that this is not the case.

Proposition 10. *If $\omega \in W^{1,2}(\Lambda M)$ and $d\omega, \delta\omega \in \Omega(M)$, then $\omega \in \Omega(M)$.*

Proof. This is clear from Corollary 9 and the fact that $d\omega, \delta\omega \in W^{k+1,2}(\Lambda M)$ implies $\omega \in W^{k,2}(\Lambda M)$. □

Corollary 11. *If $\omega \in W^{2,2}(\Lambda M)$ and $\Delta\omega \in \Omega(M)$, then $\omega \in \Omega(M)$.*

Proof. We have $d\delta d\omega = d\Delta\omega \in \Omega(M)$, $\delta d\delta\omega = \delta\Delta\omega \in \Omega(M)$. Hence, $\delta d\omega \in \Omega(M)$ and $d\delta \in \Omega(M)$ by the Proposition (recall that $d^2 = \delta^2 = 0$). Now we also obtain that $d\omega, \delta\omega \in \Omega(M)$. Using the Proposition once again, we end up with $\omega \in \Omega(M)$. □

A particular consequence of Proposition 10 is that $\mathcal{H} \subset \Omega(M)$. Therefore, it follows that $\mathcal{H}$ is the space of harmonic forms defined in Section 3. Moreover, although the operator Δ we are considering in this section acts on the larger space $W^{2,2}(\Omega)$, its kernel has not increased.

Proposition 12. *We have $\mathcal{H} = \ker \Delta$, where $\Delta : W^{2,2}(\Lambda M) \to L^2(\Omega)$ is the Laplace operator defined at the beginning of this section.*

Proof. If $\Delta\omega = 0$, Corollary 11 shows that $\omega \in \Omega(M)$. Hence, $\omega \in \mathcal{H}$ by Proposition 3. The reverse inclusion is trivial. □

Let us now see that $\mathcal{H}$ is finite-dimensional. This is a consequence of Rellich-Kondrachov's theorem.

Lemma 13. *$\mathcal{H}$ is finite-dimensional.*

Proof. Let B be the closed unit ball in $\mathcal{H}$ with respect to the $W^{1,2}$-norm. By Rellich-Kondrachov, B is a pre-compact subset of $L^2(\Lambda M)$. Now observe that B is also the closed unit ball in $\mathcal{H}$ with respect to the L^2-norm, because if $\omega \in \mathcal{H}$ then $\|\omega\|_{L^2} = \|\omega\|_{W^{1,2}}$. Hence, the closed unit ball in $\mathcal{H}$ with the L^2-norm is compact, so $\mathcal{H}$ must be finite-dimensional (recall that the closed unit ball in a normed space is compact if and only if the space is finite-dimensional). □

In particular, it follows that $\mathcal{H}$ is closed in $L^2(\Lambda M)$. Therefore, $L^2(\Lambda M)$ decomposes in orthogonal sum as

$$L^2(\Lambda M) = \mathcal{H} \oplus \mathcal{G}, \tag{16}$$

where here and in the sequel, $\mathcal{G} = L^2(\Lambda M) \ominus \mathcal{H}$ will denote the orthogonal complement of $\mathcal{H}$ in $L^2(\Lambda M)$.

We have seen that the space of harmonic functions is finite-dimensional. Our next goal is to show that Δ has closed range by proving that $\Delta|_{\mathcal{G}\cap W^{2,2}(\Lambda M)}$ is bounded below (recall that if an operator T on a Banach space is bounded below, i.e., $\|Tx\| \geq C\|x\|$, then its range is closed). The key element in the proof of this will be the next Lemma, which uses ideas from the calculus of variations. It is taken from [5, Lemma 7.3.1] and its proof is a fairly standard exercise in the calculus of variations. Here we give a self-contained proof.

Lemma 14 (Variational Lemma)**.** *Define*

$$D(\omega) = \|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2, \qquad \omega \in W^{1,2}(\Lambda M).$$

If N is a closed subspace of $L^2(\Lambda M)$, then either $N \cap W^{1,2}(\Lambda M) = 0$ or there is some $\omega_0 \in N \cap W^{1,2}(\Lambda M)$ with $\|\omega_0\|_{L^2} = 1$ and such that

$$D(\omega_0) = \inf\{D(\omega) : \omega \in N \cap W^{1,2}(\Lambda M), \|\omega\|_{L^2} = 1\}. \tag{17}$$

Proof. Assume that $N \cap W^{1,2}(\Lambda M) \neq 0$, let α be the right hand side of (17) and choose a sequence $\{\omega_n\}_{n=1}^{\infty} \subset N \cap W^{1,2}(\Lambda M)$ with $\|\omega_n\|_{L^2} = 1$ and $D(\omega_n) \to \alpha$. It follows that $\|\omega_n\|_{W^{1,2}}$ is bounded, so by Rellich-Kondrachov's theorem (see Corollary 7) there is a subsequence ω_{n_k} converging to some $\omega_0 \in W^{1,2}(\Lambda M)$ weakly in $W^{1,2}(\Lambda M)$ and strongly in $L^2(\Lambda M)$. Since N is closed in $L^2(\Lambda M)$, we have $\omega_0 \in N$. By continuity of the norm, we also get $\|\omega_0\|_{L^2} = 1$.

As soon as we can prove that if $\omega_{n_k} \to \omega_0$ weakly in $W^{1,2}(\Lambda M)$ then $D(\omega_0) \leq \liminf D(\omega_{n_k})$ (i.e., that D is sequentially weakly lower semicontinuous), then (17) will follow, thus ending the proof. Let us check this fact.

We will first prove that $d\omega_{n_k} \to d\omega_0$ weakly in $L^2(\Lambda M)$. Let $d^* : L^2(\Lambda M) \to W^{1,2}(\Lambda M)$ be the adjoint operator of d. If $\eta \in L^2(\Lambda M)$, we have

$$\langle d\omega_{n_k}, \eta\rangle_{L^2} = \langle \omega_{n_k}, d^*\eta\rangle_{W^{1,2}} \to \langle \omega_0, d^*\eta\rangle_{W^{1,2}} = \langle d\omega_0, \eta\rangle_{L^2}.$$

Hence, $d\omega_{n_k} \to d\omega_0$ weakly in $L^2(\Lambda M)$.

Now we have

$$\|d\omega_0\|_{L^2}^2 = \langle d\omega_0, d\omega_0\rangle_{L^2} = \lim_{k\to\infty} \langle d\omega_{n_k}, d\omega_0\rangle_{L^2} \leq \liminf_{k\to\infty} \|d\omega_{n_k}\|_{L^2}\|d\omega_0\|_{L^2}.$$

It follows that $\|d\omega_0\|_{L^2} \leq \liminf \|d\omega_{n_k}\|_{L^2}$. A similar reasoning proves the same inequality for δ instead of d. Therefore, $D(\omega_0) \leq \liminf D(\omega_{n_k})$. □

Lemma 15. *There is a constant $C > 0$ such that*

$$D(\omega) \geq C\|\omega\|_{W^{1,2}(\Lambda M)}^2, \qquad \omega \in \mathcal{G} \cap W^{1,2}(\Lambda M).$$

Proof. Put $N = \mathcal{G}$ and observe that it is a closed subspace of $L^2(\Lambda M)$. Let us see that $N \cap W^{1,2}(\Lambda M) \neq 0$. Take $\omega \in W^{1,2}(\Lambda M) \setminus \mathcal{H}$ and write $\omega = \omega_1 + \omega_2$, with $\omega_1 \in \mathcal{H}$, $\omega_2 \in \mathcal{G}$. It follows that $\omega_2 \in W^{1,2}(\Lambda M)$, because $\mathcal{H} \subset W^{1,2}(\Lambda M)$. Moreover, $\omega_2 \neq 0$. Hence, $N \cap W^{1,2}(\Lambda M) \neq 0$

The Variational Lemma yields an $\omega_0 \in N \cap W^{1,2}(\Lambda M)$ with $\|\omega_0\|_{L^2} = 1$ and satisfying (17). By linearity, we have

$$D(\omega) \geq D(\omega_0)\|\omega\|_{L^2}^2, \qquad \omega \in N \cap W^{1,2}(\Lambda M).$$

We also have $D(\omega_0) > 0$, because $\omega_0 \notin \mathcal{H}$. This together with $\|\omega\|_{W^{1,2}}^2 = \|\omega\|_{L^2}^2 + D(\omega)$ implies the Lemma, with $C = (D(\omega_0)^{-1} + 1)^{-1}$. □

Now we are ready to show that $\Delta|_{\mathcal{G}\cap W^{2,2}(\Lambda M)}$ is bounded below.

Proposition 16. *There is a constant $C > 0$ such that*

$$\|\Delta\omega\|_{L^2}^2 \geq C\|\omega\|_{W^{2,2}}^2, \qquad \omega \in \mathcal{G} \cap W^{2,2}(\Lambda M).$$

Proof. Let $\omega \in \mathcal{G} \cap W^{2,2}(\Lambda M)$. Since $d(W^{1,2}(\Lambda M)) \perp \delta(W^{1,2}(\Lambda M))$ and $d^2 = \delta^2 = 0$, we have

$$\|\Delta\omega\|_{L^2}^2 = \|d\delta\omega\|_{L^2}^2 + \|\delta d\omega\|_{L^2}^2 = D(\delta\omega) + D(d\omega).$$

Note that $d\omega, \delta\omega \in \mathcal{G}$. This is proved as in the beginning of Section 4.1. Applying twice the preceding lemma, we get

$$\begin{aligned} D(\delta\omega) + D(d\omega) &\geq C[\|\delta\omega\|_{W^{1,2}}^2 + \|d\omega\|_{W^{1,2}}^2] \geq \frac{C}{2}[\|\delta\omega\|_{W^{1,2}}^2 + \|d\omega\|_{W^{1,2}}^2] + \frac{C}{2}D(\omega) \\ &\geq \frac{C}{2}[\|\delta\omega\|_{W^{1,2}}^2 + \|d\omega\|_{W^{1,2}}^2] + \frac{C^2}{2}\|\omega\|_{L^2}^2 \geq C'\|\omega\|_{W^{2,2}}^2, \end{aligned}$$

where $C' = \min\{C, C^2\}/2$. The Proposition now follows. □

Using this Proposition we can easily prove that Δ has closed range.

Proposition 17. *$\Delta(W^{2,2}(\Lambda M))$ is closed.*

Proof. Let us first see that

$$\Delta(W^{2,2}(\Lambda M)) = \Delta(\mathcal{G} \cap W^{2,2}(\Lambda M)). \tag{18}$$

Let $\omega \in W^{2,2}(\Lambda)$. Using (16), we have $\omega = \omega_1 + \omega_2$, with $\omega_1 \in \mathcal{H}$, $\omega_2 \in \mathcal{G}$. Since $\mathcal{H} \subset \Omega(M) \subset W^{2,2}(\Lambda M)$, it follows $\omega_2 \in \mathcal{G} \cap W^{2,2}(\Lambda M)$. Proposition 12 implies that $\Delta\omega = \Delta\omega_2$. Hence, we have (18).

Now observe that $\mathcal{G}' = \mathcal{G} \cap W^{2,2}(\Lambda M)$ is closed in $W^{2,2}(\Lambda M)$. This is because the inclusion $J : W^{2,2}(\Lambda M) \to L^2(\Lambda M)$ is continuous, $\mathcal{G}' = J^{-1}(\mathcal{G})$ and $\mathcal{G}$ is closed in $L^2(\Lambda M)$.

Consider the restriction $\Delta|_{\mathcal{G}'}$. By proposition 16, this operator is bounded below, so its range is closed. Now (18) proves that the range of $\Delta|_{\mathcal{G}'}$ coincides with that of Δ. □

The operator Δ on $W^{2,2}(\Lambda M)$ is not self-adjoint, since even its domain and codomain are different. Thus, we will also need to study the adjoint operator Δ^*. However, it turns out that its kernel coincides with that of Δ. To prove this, we first need to show that $\ker \Delta^* \subset \Omega(M)$. Indeed, it would suffice to prove $\ker \Delta^* \subset W^{2,2}(\Lambda M)$ (see the proof of Lemma 19 below). Although proving that $\ker \Delta \subset \Omega(M)$ was easy (see Corollary 11), the analogous result for Δ^* requires the theory of distributions and regularity of elliptic differential operators. Hence, we will only give a rough sketch of the proof of this fact.

Lemma 18. *We have* $\ker \Delta^* \subset \Omega(M)$.

To prove this, one observes that $\Delta^* \omega = 0$ implies that

$$\langle \omega, \Delta\eta \rangle_{L^2} = 0, \qquad \forall \eta \in \Omega(M). \tag{19}$$

Hence, we have $\Delta\omega = 0$ in the sense of distributions (because Δ symmetric). Now one can prove that the operator Δ is elliptic (see, for instance, [8, Section 6.33]) and then apply an elliptic regularity theorem to obtain that $\omega \in \Omega(M)$ because $0 \in \Omega(M)$ (see [6, Theorem 8.12] or [8, Theorem 6.30]).

Also note that this lemma follows directly from [8, Theorem 6.5]. Indeed, using the terminology there, (19) implies that ω (as a functional acting on $\Omega(M)$) is a weak solution of $\Delta\omega = 0$.

Lemma 19. *We have* $\ker \Delta^* = \ker \Delta = \mathcal{H}$.

Proof. First note that

$$\langle \Delta\omega, \eta \rangle_{L^2} = \langle \omega, \Delta\eta \rangle_{L^2}, \qquad \omega, \eta \in W^{2,2}(\Lambda M).$$

The proof of this is the same as that of Proposition 2, but using Proposition 5 instead of Proposition 1.

Assume that $\omega \in \ker \Delta^*$. Then $\omega \in W^{2,2}(\Lambda M)$ by Lemma 18. If $\eta \in W^{2,2}(\Lambda M)$, we have

$$0 = \langle \Delta^* \omega, \eta \rangle_{W^{2,2}} = \langle \omega, \Delta\eta \rangle_{L^2} = \langle \Delta\omega, \eta \rangle_{L^2},$$

so that $\omega \in \ker \Delta$, because $W^{2,2}(\Lambda M)$ is dense in $L^2(\Lambda M)$. This shows that $\ker \Delta^* \cap W^{2,2}(\Lambda M) \subset \ker \Delta$. The reverse inclusion is obtained by tracing backwards the same steps. □

We can now finish the proof of the Hodge theorem.

Proof of Theorem 4. We start with the decomposition

$$L^2(\Lambda M) = \ker \Delta^* \oplus (\ker \Delta^*)^\perp = \ker \Delta^* \oplus \overline{\Delta(W^{2,2}(\Lambda M))} = \ker \Delta \oplus \Delta(W^{2,2}(\Lambda M)).$$

Here we have used Proposition 17 and Lemma 19 in the last equality.

Now, if $\omega \in \Omega(M)$, we write $\omega = \omega_1 + \omega_2$, with $\omega_1 \in \ker \Delta$ and $\omega_2 \in \Delta(W^{2,2}(\Lambda M))$. Since $\ker \Delta \subset \Omega(M)$, it follows that $\omega_2 \in \Omega(M)$. Using Corollary 11, we get $\omega_2 \in \Delta(\Omega(M))$. This proves the Theorem. □

7. Green's operator

In order to give an application of the Hodge theorem to cohomology, we will first need to introduce Green's operator G. Recall the definition of $\mathcal{H}$ and $\mathcal{G}$ from (15) and (16), and that $\mathcal{H}$ coincides with the space of harmonic forms $\ker \Delta$ that appears in (12). The Hodge theorem implies that

$$\mathcal{G} \cap \Omega(M) = \Delta(\Omega(M)).$$

Put $\mathcal{K} = \Delta(\Omega(M))$, considered with the L^2-norm (i.e., the one defined on (7)), so that

$$\Omega(M) = \mathcal{H} \oplus \mathcal{K}.$$

The restriction $\Delta|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ is surjective, because $\Delta|_{\mathcal{H}} = 0$, and bounded below, because of Proposition 16 (recall that $\Omega(M) \subset W^{2,2}(\Lambda M)$ and that $\|\omega\|_{W^{2,2}} \geq \|\omega\|_{L^2}$). Hence, $(\Delta|_{\mathcal{K}})^{-1}$ is a bounded operator on $\mathcal{K}$.

Let $P_{\mathcal{H}}$ be the orthogonal projection onto $\mathcal{H}$. We define $G : \Omega(M) \to \mathcal{K}$ by

$$G = (\Delta|_{\mathcal{K}})^{-1}(I - P_{\mathcal{H}}). \tag{20}$$

Note that this is well defined because $(I - P_{\mathcal{H}})\Omega(M) = \mathcal{K}$. We immediately obtain

$$\Delta G\omega = \omega - P_{\mathcal{H}}\omega, \qquad \omega \in \Omega(M). \tag{21}$$

Therefore, it is easy to see that, for each $\omega \in \Omega(M)$, $\eta = G\omega$ is the unique solution in $\mathcal{K}$ of the problem

$$\Delta\eta = \omega - P_{\mathcal{H}}\omega.$$

This explains the name "Green's operator", by analogy with the theory of Green's operators for PDEs.

Now we will prove a few basic properties of Green's operator that are important on its own but will not be needed in the sequel.

Proposition 20. *G is symmetric, i.e.,*

$$\langle G\omega, \eta\rangle = \langle \omega, G\eta\rangle, \qquad \omega, \eta \in \Omega(M).$$

Proof. Just use (21) to compute

$$\langle G\omega, \eta\rangle = \langle G\omega, \eta - P_{\mathcal{H}}\eta\rangle = \langle G\omega, \Delta G\eta\rangle = \langle \Delta G\omega, G\eta\rangle = \langle \omega - P_{\mathcal{H}}\omega, G\eta\rangle = \langle \omega, G\eta\rangle. \qquad \square$$

Proposition 21. *G is a compact operator.*

Proof. By Proposition 16 and (18), $\Delta : \mathcal{G} \cap W^{2,2}(\Lambda M) \to \Delta(W^{2,2}(\Lambda M))$ is surjective and bounded below, hence invertible (here we are using the $W^{2,2}$-norm in the domain and the L^2-norm in the codomain). Therefore, the operator

$$T = (\Delta|_{\mathcal{G}\cap W^{2,2}(\Lambda M)})^{-1} : \Delta(W^{2,2}(\Lambda M)) \to \mathcal{G} \cap W^{2,2}(\Lambda M)$$

is bounded when we put the L^2-norm in the domain and the $W^{2,2}$-norm in the codomain.

Rellich-Kondrachov (see Theorem 6) implies that T is compact if we put the L^2-norm on both the domain and codomain. The operator $(\Delta|_{\mathcal{K}})^{-1}$ is just a restriction of T, so the compactness of G follows. $\square$

The following property of G will be used below.

Proposition 22. *G commutes with any operator which commutes with Δ. In particular, it commutes with d and δ.*

Proof. Assume that T commutes with Δ. Then T commutes with $(\Delta|_{\mathcal{K}})^{-1}$. It suffices to see that T also commutes with $P_{\mathcal{H}}$.

Note that $T\Delta = \Delta T$ implies $T\mathcal{H} \subset \mathcal{H}$ and $T\mathcal{K} \subset \mathcal{K}$. Hence,

$$P_{\mathcal{H}}T = P_{\mathcal{H}}TP_{\mathcal{H}} + P_{\mathcal{H}}T(I - P_{\mathcal{H}}) = P_{\mathcal{H}}TP_{\mathcal{H}} = TP_{\mathcal{H}},$$

so that T commutes with $P_{\mathcal{H}}$. $\square$

8. An application to the de Rham cohomology

The following theorem is the main application of the Hodge theorem to the de Rham cohomology. We will also use it to prove the Poincaré duality for the de Rham cohomology.

Theorem 23. *If M is a compact oriented Riemannian manifold, there is a unique harmonic representant in each de Rham cohomology class of M.*

Proof. Assume that $d\omega = 0$. Then by (21) and Proposition 22,

$$\omega = \Delta G\omega + P_{\mathcal{H}}\omega = d\delta G\omega + \delta dG\omega + P_{\mathcal{H}}\omega = d\delta G\omega + \delta Gd\omega + P_{\mathcal{H}}\omega = d\delta G\omega + P_{\mathcal{H}}\omega.$$

This shows that $P_{\mathcal{H}}\omega$ is an harmonic form in the same cohomology class as ω.

Now assume that ω_1, ω_2 are harmonic and $\omega_1 - \omega_2 = d\eta$. Then, by Proposition 3,

$$\|\omega_1 - \omega_2\|^2 = \langle d\eta, \omega_1 - \omega_2\rangle = \langle \eta, \delta\omega_1 - \delta\omega_2\rangle = 0.$$

Hence, $\omega_1 = \omega_2$. $\square$

Let us finish by proving the Poincaré duality for the de Rham cohomology. Assume that M is a compact oriented differentiable manifold of dimension n. Recall that Stoke's theorem shows that the bilinear form

$$B([\omega],[\eta]) = \int_M \omega \wedge \eta, \qquad [\omega] \in H^k(M),\ [\eta] \in H^{n-k}(M)$$

is well defined, i.e., it does not depend on the choice of representants for the cohomology classes. We will show now that B is non-singular (meaning that for each $[\omega] \in H^k(M)$, $[\omega] \neq 0$, there is some $[\eta] \in H^{n-k}(M)$ such that $B([\omega],[\eta]) \neq 0$). Once we prove this, it will follow that B gives a duality pairing between $H^k(M)$ and $H^{n-k}(M)$, so that $(H^k(M))^* \cong H^{n-k}(M)$.

To do this, first choose a Riemmanian metric on M (this can be done locally and then pasting with partitions of unity or by embedding M in $\mathbb{R}^m$ and giving it the induced metric).

Now take $[\omega] \in H^k(M)$, $[\omega] \neq 0$. Let $\omega \neq 0$ be the harmonic representant of $[\omega]$. Since $*\Delta = \Delta *$ (this is easy to check using the definitions), $*\omega$ is harmonic, so it defines a cohomology class $[*\omega] \in H^{n-k}(M)$. Now

$$B([\omega],[*\omega]) = \int_M \omega \wedge *\omega = \|\omega\|^2 > 0,$$

by (9). It follows that B is non-singular.

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