Separation of singularities, generation of algebras and complete K-spectral sets

Daniel Estévez

Universidad Autónoma de Madrid

Joint work with Michael Dritschel (Newcastle Univ.) and Dmitry Yakubovich (UAM)

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Summary

1. Test collections and complete K-spectral sets
2. Separation of singularities
3. Generation of algebras
4. Fitting everything together: idea of the proofs of the results about test collections

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3. Generation of algebras
4. Fitting everything together: idea of the proofs of the results about test collections

Von Neumann's inequality

If T is a contraction on a Hilbert space H (i.e., $\|T\| \leq 1$), then

$$\|p(T)\| \leq \max_{z \in \overline{\mathbb{D}}} |p(z)|,$$

for every polynomial p.

In fact,

$$\|f(T)\|_{\mathcal{B}(H^s)} \leq \max_{z \in \overline{\mathbb{D}}} \|f(z)\|,$$

for every for every rational function $f = [f_{jk}]_{j,k=1}^{s}$ with values on $s \times s$ matrices and no poles in X, and every $s \geq 1$.

Here, $f(T) = [f_{jk}(T)]_{j,k=1}^{s}$.

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Complete K-spectral sets

Definition

H a Hilbert space, $T \in \mathcal{B}(H)$ a bounded operator, $X \subset \widehat{\mathbb{C}}$ a compact set. X is a complete K-spectral set for T if

$$\|f(T)\|_{\mathcal{B}(H^s)} \leq K \max_{z \in X} \|f(z)\|_{\mathcal{B}(\mathbb{C}^s)},$$

for every rational function $f = [f_{jk}]_{j,k=1}^s$ with values on $s \times s$ matrices and no poles in X, and every $s \geq 1$.

- T is a contraction if and only if $\overline{\mathbb{D}}$ is a complete 1-spectral set (von Neumann's inequality).
- T is similar to a contraction ($T = SAS^{-1}$, $\|A\| \leq 1$) if and only if $\overline{\mathbb{D}}$ is a complete K-spectral set for some K.
- $X = \overline{\Omega}$, Ω simply-connected. $\varphi : \mathbb{D} \to \Omega$ the Riemann map. X is complete K-spectral for T if and only if $T = S\varphi(A)S^{-1}$, $\|A\| \leq 1$
- T is similar to an operator having a rational normal dilation to ∂X if and only if X is a complete K-spectral set for some K. This means that there is $\widetilde{H} \supset H$ and $N \in \mathcal{B}(\widetilde{H})$ normal with $\sigma(N) \subset \partial X$ such that

$$Sf(T)S^{-1} = P_H f(N)|H, \qquad \forall f \text{ rational with no poles on } X.$$

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Some results about complete K-spectral sets

1. Let $\Omega_1, \ldots, \Omega_n \subset \widehat{\mathbb{C}}$ be simply connected domains with analytic boundaries and such that their boundaries do not intersect. Then $\overline{\bigcap \Omega_j}$ is complete K-spectral for T if and only if $\overline{\Omega_j}$ is complete K_j-spectral for T. *(Douglas, Paulsen, 1986)*.
2. Let $D_1, \ldots, D_n$ be discs in $\widehat{\mathbb{C}}$. If $\overline{D_j}$ is (complete) 1-spectral for T, then $\overline{\bigcap D_j}$ is complete K-spectral for T. *(Badea, Beckermann, Crouzeix, 2009)*.
3. Let X be a compact convex set. If the numerical range of T

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$$

is contained in X, then X is a complete K-spectral set for T. *(Delyon, Delyon, 1999)*.
4. Let B be a finite Blaschke product. If $\sigma(T) \subset \overline{\mathbb{D}}$ and $\overline{\mathbb{D}}$ is complete K'-spectral for $B(T)$, then $\overline{\mathbb{D}}$ is complete K-spectral for T. *(Mascioni, 1994)*.

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Some of our generalizations of these results

Theorem

Let $\Omega_1, \ldots, \Omega_s$ be Jordan domains with rectifiable and Ahlfors regular boundaries that intersect transversally. If $\overline{\Omega}_j$ is (complete) K_j-spectral for T, then $\overline{\bigcap \Omega_j}$ is (complete) K-spectral for T.

Theorem

Let Ω be a Jordan domain with $C^{1,\alpha}$ boundary. If $\overline{\Omega}$ and $\mathbb{C} \setminus \overline{\Omega}$ are K-spectral for T, then $\partial\Omega$ is complete K'-spectral for T. Hence, T is similar to a normal operator with spectrum in $\partial\Omega$.

Theorem

Let Ω be a Jordan domain and $R > 0$ such that for each $\lambda \in \Omega$ there is $\mu \in \mathbb{C} \setminus \overline{\Omega}$ such that $B(\mu, R)$ is tangent to $\partial\Omega$ at λ. If $\|(T - \mu I)^{-1}\| \leq R^{-1}$, then $\overline{\Omega}$ is complete K-spectral for some $K > 0$.

If $\sigma(T) \subset \Gamma$ and $\|(T - zI)^{-1}\| \leq \operatorname{dist}(z, \Gamma)^{-1}$, then T is normal *(Stampfli, 1969)*.

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Our main problem:

$X \subset \widehat{\mathbb{C}}$ some set. We look for a collection Φ of functions analytic in X such that

$$\sigma(T) \subset X,\ \|\varphi(T)\| \leq 1, \forall \varphi \in \Phi \Rightarrow \overline{X} \text{ is complete } K\text{-spectral for } T, \qquad (*)$$

or

$$\begin{gathered}\sigma(T) \subset X, \overline{\mathbb{D}} \text{ is complete } K'\text{-spectral for } \varphi(T), \forall \varphi \in \Phi \Rightarrow \\ \overline{X} \text{ is complete } K\text{-spectral for } T.\end{gathered} \qquad (**)$$

- Tipically, $X = \Omega$ an open domain, or $X = \overline{\Omega}$.

Definition

- Φ is a test collection in X if $(*)$ holds, with $K = K(\Omega, \Phi)$.
- Φ is a strong test collection in X if $(**)$ holds, with $K = K(\Omega, \Phi, K')$.
- Φ is a non-uniform test collection in X if $(*)$ holds, with $K = K(\Omega, \Phi, T)$.
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Test collections

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1. Let $\Omega_1, \dots, \Omega_n \subset \widehat{\mathbb{C}}$ be simply connected domains with analytic boundaries and such that their boundaries do not intersect. Let $\varphi_k : \overline{\Omega_k} \to \overline{\mathbb{D}}$ be Riemann conformal mappings. Then $\{\varphi_1, \dots, \varphi_n\}$ is a strong test collection in $\overline{\bigcap \Omega_k}$. *(Douglas, Paulsen, 1986)*.
2. Let $D_1, \dots, D_n$ be discs in $\widehat{\mathbb{C}}$. Let φ_k be a Möbius transformation taking D_k onto $\mathbb{D}$. Then $\{\varphi_1, \dots, \varphi_n\}$ is a test collection in $\overline{\bigcap D_k}$. *(Badea, Beckermann, Crouzeix, 2009)*.
3. Let X be a compact convex set. Write $X = \bigcap H_\alpha$, with H_α closed half-planes. Let φ_α be a Möbius transformation taking H_α onto $\overline{\mathbb{D}}$. Then $\{\varphi_\alpha\}$ is a test collection in X. *(Delyon, Delyon, 1999)*.
4. If B is a finite Blaschke product, the set $\{B\}$ is a non-uniform strong test collection in $\overline{\mathbb{D}}$. *(Mascioni, 1994)*.

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Previous results restated in the language of test collections

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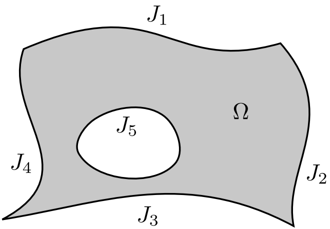
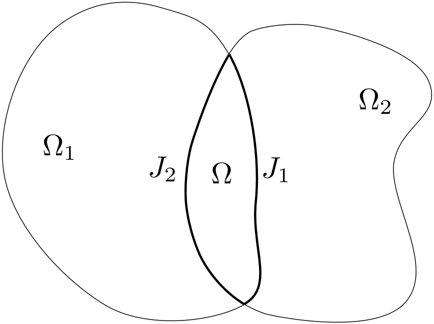
Definition

- $\Omega \subset \mathbb{C}$ a domain such that $\partial\Omega$ is a disjoint finite union of piecewise analytic Jordan curves. We assume that the interior angles of the “corners” of $\partial\Omega$ are between 0 and π.
- $\{J_k\}_{k=1}^n$ closed analytic arcs intersecting each other at most in two points and such that $\partial\Omega = \bigcup J_k$.
- $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ analytic in $\overline{\Omega}$ (can be weakened in many cases).
- $|\varphi_k| = 1$ in J_k.
- φ_k' does not vanish in J_k.
- $\varphi_k(\zeta) \neq \varphi_k(z)$ if $\zeta \in J_k$, $z \in \overline{\Omega}$, and $z \neq \zeta$.



Example

$\Omega_1, \ldots, \Omega_n$ simply connected domains with analytic boundaries and such that their boundaries intersect transversally.
$\Omega = \bigcap \Omega_k$, $J_k = \partial\Omega \cap \partial\Omega_k$.
$\varphi_k : \overline{\Omega_k} \to \overline{\mathbb{D}}$ Riemann conformal mappings.



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A simple example of an admissible map

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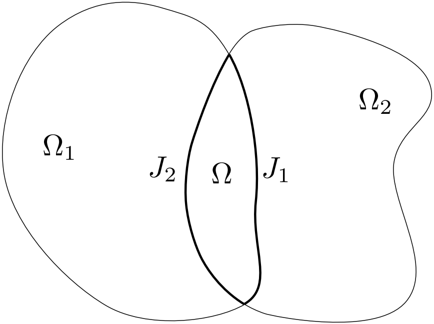

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Our results about admissible maps

Theorem A

Let Ω be a simply connected domain, and $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ admissible. Then Φ is a non-uniform strong test collection in $\overline{\Omega}$. If Φ is injective and Φ' does not vanish in Ω, then Φ is a strong test collection in $\overline{\Omega}$.

Theorem B

Let Ω be a not necessarily simply connected domain. If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and injective and Φ' does not vanish in Ω, then Φ is a strong test collection in Ω.

Summary

A simple case of an admissible function

Let Ω_1, Ω_2 be simply connected domains whose boundaries are analytic and intersect transversally. Put $\Omega = \Omega_1 \cap \Omega_2$.
Let $\varphi_k : \Omega_k \to \mathbb{D}$ be Riemann conformal mappings.
Then $\Phi = (\varphi_1, \varphi_2) : \overline{\Omega} \to \overline{\mathbb{D}}^2$ is admissible.

To prove that Φ is a test collection we can use a decomposition of $f \in H^\infty(\Omega)$ as

$$f = g_1 \circ \varphi_1 + g_2 \circ \varphi_2,$$

with $g_k \in H^\infty(\mathbb{D})$.

We denote $f_k = g_k \circ \varphi_k$. The problem is equivalent to writing

$$f = f_1 + f_2,$$

with $f_k \in H^\infty(\Omega_k)$, because $\exists \varphi_k^{-1}$.

How to decompose $f = f_1 + f_2$?

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$$f = g_1 \circ \varphi_1 + g_2 \circ \varphi_2,$$

with $g_k \in H^\infty(\mathbb{D})$.

We denote $f_k = g_k \circ \varphi_k$. The problem is equivalent to writing

$$f = f_1 + f_2,$$

with $f_k \in H^\infty(\Omega_k)$, because $\exists \varphi_k^{-1}$.

How to decompose $f = f_1 + f_2$?

Let Ω_1, Ω_2 be simply connected domains whose boundaries intersect transversally. Put $\Omega = \Omega_1 \cap \Omega_2$. Then $f \in H^\infty(\Omega)$ can be written as

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f has singularities in $\partial\Omega$, f_j has singularities in $\partial\Omega \cap \partial\Omega_j$. The singularities of f have been separated somehow.

How to do this? *First try (wrong).* Write f as its Cauchy integral $f = \mathcal{C}_{\partial\Omega} f$. Put $J_k = \partial\Omega \cap \partial\Omega_k$, and $f_k = \mathcal{C}_{J_k} f$. Then $f = f_1 + f_2$, but f_k have logarithmic singularities in the points of $J_1 \cap J_2$, so $f_k \notin H^\infty(\Omega_k)$. (This would have worked for H^p, $p < \infty$, instead of H^∞).

The idea of H-N-OC. Put $\{z_1, z_2\} = J_1 \cap J_2$. Put $\Gamma_k = J_k \cap \mathbb{D}_\varepsilon(z_k)$. Let R_k be a rigid rotation around z_k such that $R_k(\Gamma_k)$ is outside Ω. Then they show that

$$f_1 = \mathcal{C}_{J_1} f + \mathcal{C}_{R_2(\Gamma_2)}(f \circ R_2^{-1}) - \mathcal{C}_{R_1(\Gamma_1)}(f \circ R_1^{-1}) \in H^\infty(\Omega_1),$$

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Separation of singularities with the composition

If φ_k are univalent, we have seen how to write

$$f = g_1 \circ \varphi_1 + g_2 \circ \varphi_2$$

by putting $g_k = f_k \circ \varphi_k^{-1}$.

What can we do if $\varphi_k : \Omega \to \mathbb{D}$ are not univalent, but they still send J_k bijectively onto some arc of $\mathbb{T}$?

Our main analytic tool:

Theorem

Let Ω and $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be admissible. Then there exist bounded linear operators $F_k : H^\infty(\Omega) \to H^\infty(\mathbb{D})$ such that the operator

$$f \mapsto f - \sum_{k=1}^{n} F_k(f) \circ \varphi_k$$

is compact in $H^\infty(\Omega)$ and its range is contained in $A(\overline{\Omega}) = \mathrm{Hol}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Moreover, F_k map $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$.

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Techniques of the proof

- The integral operator

$$f \mapsto \int_{J_k} \left[\frac{1}{\zeta - z} - \frac{\varphi_k'(\zeta)}{\varphi_k(\zeta) - \varphi_k(z)} \right] f(\zeta)\, d\zeta$$

  is weakly singular. Hence compact.

- Replace the Cauchy integrals

$$\int_{J_k} \frac{1}{\zeta - z} f(\zeta)\, d\zeta$$

  by *modified* Cauchy integrals

$$\int_{J_k} \frac{\varphi_k'(\zeta)}{\varphi_k(\zeta) - z} f(\zeta)\, d\zeta,$$

  which are analytic in $\mathbb{C} \setminus \varphi_k(J_k)$.

- Use the trick of Havin–Nersessian–Ortega-Cerdá to get functions in $H^\infty(\mathbb{D})$ when *cutting* f into a sum of Cauchy integrals in arcs J_k. $f = \mathcal{C}_{\partial\Omega} f = \sum \mathcal{C}_{J_k} f$.

Summary

Ω some domain, $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$.

$$\mathcal{H}_\Phi = \left\{ \sum_{j=1}^{l} f_{j,1}(\varphi_1(z)) f_{j,2}(\varphi_2(z)) \cdots f_{j,n}(\varphi_n(z)) : \; l \in \mathbb{N}, \; f_{j,k} \in H^\infty(\mathbb{D}) \right\}$$

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These are the (non-closed) subalgebras of $H^\infty(\Omega)$ and $A(\overline{\Omega})$ generated by functions of the form $f \circ \varphi_k$, with $f \in H^\infty(\mathbb{D})$ or $f \in A(\overline{\mathbb{D}})$.

Questions:

- What geometric conditions on Φ guarantee that $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$?
- What geometric conditions on Φ guarantee that $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$ are closed subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\overline{\Omega})$ (respectively)?

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Finite codimension

Theorem

If Ω and Φ are admissible, then $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$ are closed subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\overline{\Omega})$ respectively.

Proof.

Put $Gf = \sum_{k=1}^{n} F_k(f) \circ \varphi_k$. Then $G : H^\infty(\Omega) \to H^\infty(\Omega)$ and $G - I$ is compact. Hence, $GH^\infty(\Omega)$ is a closed subspace of finite codimension in $H^\infty(\Omega)$. Note that $GH^\infty(\Omega) \subset \mathcal{H}_\Phi$.

For $\mathcal{A}_\Phi$, use the restriction $G|A(\overline{\Omega})$. □

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Equalities $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$

Theorem

If Ω and Φ are admissible, Φ is injective in $\overline{\Omega}$, and Φ' does not vanish in Ω, then $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$.

Note: It is easy to see that Φ being injective and Φ' not vanishing are necessary conditions for the equalities to hold.

The proof uses Banach algebra tools and the following classification of the one-codimensional closed unital subalgebras A_0 of a unital Banach algebra A (Gorin, 1969).

A_0 can have one of the following two forms:

- $A_0 = \ker(\psi_1 - \psi_2)$, where $\psi_1, \psi_2 \in \mathfrak{M}(A)$, $\psi_1 \neq \psi_2$. (Informally, A_0 are the functions which coincide at the points ψ_1 and ψ_2).
- $A_0 = \ker \eta$, where $\eta \neq 0$ is a continuous derivation at some $\psi \in \mathfrak{M}(A)$, i.e., $\eta \in A^*$ and

$$\eta(fg) = \eta(f)\psi(g) + \psi(f)\eta(g), \qquad \forall f, g \in A.$$

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$\mathcal{V} = \Phi(\Omega)$ is an analytic curve in the polydisc $\mathbb{D}^n$. We consider the algebras $H^\infty(\mathcal{V})$ and $A(\overline{\mathcal{V}})$.
Put $\Phi^* F = F \circ \Phi$.

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If Ω and Φ are admissible, then $\Phi^* H^\infty(\mathcal{V}) = \mathcal{H}_\Phi$ and $\Phi^* A(\overline{\mathcal{V}}) = \mathcal{A}_\Phi$.

The Agler algebra of $\mathbb{D}^n$:

$$\|f\|_{\mathcal{SA}(\mathbb{D}^n)} = \sup_{\substack{\|T_j\| \leq 1 \\ \sigma(T_j) \subset \mathbb{D}}} \|f(T_1, \ldots, T_n)\|.$$

For every n, $\mathcal{SA}(\mathbb{D}^n) \subset H^\infty(\mathbb{D}^n)$. For $n = 1, 2$, there is equality, but for $n \geq 3$, it is believed that the inclusion is proper.

Theorem

If Ω and Φ are admissible, then every $f \in H^\infty(\mathcal{V})$ can be extended to an $F \in \mathcal{SA}(\mathbb{D}^n)$ with $\|F\|_{\mathcal{SA}(\mathbb{D}^n)} \leq C\|f\|_{H^\infty(\mathcal{V})}$.

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Theorem B

If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and injective and Φ' does not vanish in Ω, then Φ is a strong test collection in Ω.

Take T with $\sigma(T) \subset \Omega$ and such that $\overline{\mathbb{D}}$ is complete K-spectral for $\varphi_k(T)$, and f a $s \times s$-matrix–valued rational function with no poles in $\overline{\Omega}$. We must show that

$$\|f(T)\| \leq C \max_{z \in \overline{\Omega}} \|f(z)\|.$$

We do the case $s = 1$.
Put $Gf = \sum F_k(f) \circ \varphi_k$. Since $G - I$ is compact, there exist an operator R and an operator P with finite-dimensional range such that $I = GR + P$. We can write

$$f = \sum_{k=1}^{n} F_k(Rf) \circ \varphi_k + \sum_{j=1}^{r} \alpha_j(f) g_j,$$

where $\alpha_j \in (A(\overline{\Omega}))^*$ and $g_j \in A(\overline{\Omega}) = \mathcal{A}_\Phi$.

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$$\|f(T)\| \leq \sum_{k=1}^{n} \|F_k(Rf)(\varphi_k(T))\| + \sum_{j=1}^{r} |\alpha_j(f)| \|g_j(T)\| \leq C\|f\|_\infty.$$

The case $s \geq 2$ is the same. We have to use that an operator whose range is contained in a commutative C^*-algebra is automatically completely bounded. This means that the bounds that we have obtained before are uniform in s.

Theorem A

Let Ω be a simply connected domain, $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ admissible. Then Φ is a non-uniform strong test collection in $\overline{\Omega}$. If Φ is injective and Φ' does not vanish in Ω, then Φ is a strong test collection in $\overline{\Omega}$.

Here $\sigma(T)$ can intersect $\partial\Omega$. We cannot use the previous argument.

Idea: To use a *shrinking* of Ω.

- $\{\psi_\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0}$ analytic and univalent functions on some open $U \supset \overline{\Omega}$.
- $\psi_0 \equiv z$.
- $\psi_\varepsilon(\overline{\Omega}) \subset \Omega$ for $\varepsilon > 0$.
- $\varepsilon \mapsto \psi_\varepsilon$ is continuous in the topology of uniform convergence on compact subsets of U.

To construct the shrinking we need that Ω is simply connected.

- Pass to operators $T_\varepsilon = \psi_\varepsilon(T)$.
- $\sigma(T_\varepsilon) \subset \Omega$.
- $T_\varepsilon \to T$ in operator norm.

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Thank you!