K-spectral sets, operator tuples and related function theory

Daniel Estévez Sánchez

PhD thesis defense
Advisor: Dmitry Yakubovich

Universidad Autónoma de Madrid

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Outline

1. Part I: Test functions and K-spectral sets
   - Test collections
   - Subalgebras of analytic functions
   - An application: operators with spectra on a curve

2. Part II: Separating structures and operator tuples
   - Separating structures
   - Vessels and generalized compression

Outline

K-spectral sets

- If T is a contraction ($\|T\| \leq 1$) in a Hilbert space, then

$$\|p(T)\| \leq \sup_{|z| \leq 1} |p(z)|$$

  for every polynomial p (von Neumann's inequality).
- If T is similar to a contraction ($\|VTV^{-1}\| \leq 1$),

$$\|p(T)\| \leq K \sup_{|z| \leq 1} |p(z)|.$$

- Generalization to domains different from $\overline{\mathbb{D}}$: $X \subset \mathbb{C}$ compact, $\sigma(T) \subset X$. Then X is *K-spectral* for T if

$$\|f(T)\| \leq K \sup_{z \in X} |f(z)| \qquad (*)$$

  for every rational function f with poles off X.
- If $(*)$ holds for every $s \times s$ matrix-valued rational function f with poles off X, for every $s \in \mathbb{N}$, and with K independent of s, then X is called *complete K-spectral*.
- $\overline{\mathbb{D}}$ is complete K-spectral for T if and only if T is similar to a contraction.

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Compression and dilation

If $A \in \mathcal{B}(K)$, K decomposes as

$$K = G_- \oplus H \oplus G_+$$

and A has the structure

$$A = \begin{bmatrix} * & 0 & 0 \\ * & B & 0 \\ * & * & * \end{bmatrix},$$

then B is called a *compression* of A and A is called a *dilation* of B.

For every polynomial p,

$$p(A) = \begin{bmatrix} * & 0 & 0 \\ * & p(B) & 0 \\ * & * & * \end{bmatrix}.$$

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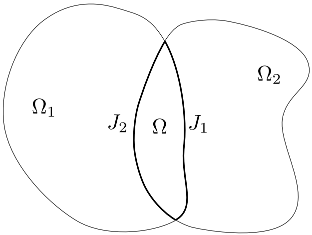
Test functions: a simple example

- $\Omega_1, \Omega_2 \subset \mathbb{C}$ Jordan domains whose boundaries intersect transversally, $\Omega = \Omega_1 \cap \Omega_2$, $\varphi_j : \Omega_j \to \mathbb{D}$ the Rieman mappings
- **Result:** If $\sigma(T) \subset \Omega$ and $\|\varphi_j(T)\| \leq 1$, then $\overline{\Omega}$ is K-spectral for T, with K independent of T
- **Proof:** If $f \in A(\overline{\Omega})$, then $f = f_1 + f_2$ with $f_j \in A(\overline{\Omega}_j)$ and $\|f_j\|_\infty \leq C\|f\|_\infty$ (Havin-Nersessian separation of singularities). Then

$$\|f(T)\| = \|(f_1 \circ \varphi_1^{-1})(\varphi_1(T)) + (f_2 \circ \varphi_2^{-1})(\varphi_2(T))\| \leq \|f_1 \circ \varphi_1^{-1}\|_\infty + \|f_2 \circ \varphi_2^{-1}\|_\infty \leq 2C\|f\|_\infty$$

- To prove complete K-spectrality, we need an additional lema about C^*-alebgras.
- Try to extend this result to a more general situation (φ_j not univalent)



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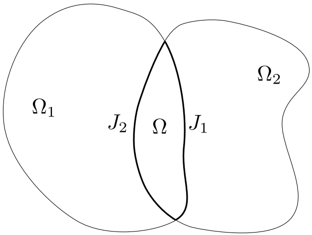

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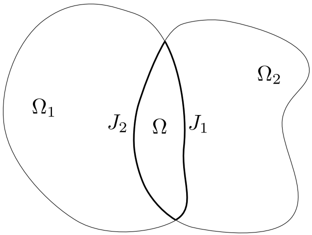


Test collections

Definition

Let $X \subset \widehat{\mathbb{C}}$ and Φ a collection of functions taking X into $\overline{\mathbb{D}}$, analytic in neighbourhoods of X. We say that Φ is a *test collection* over X if

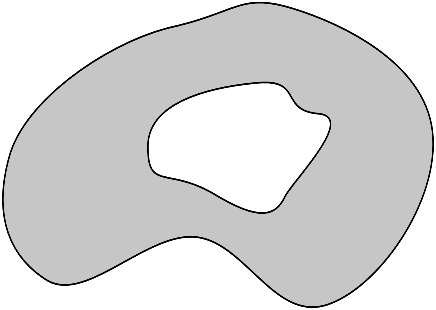
$$\overline{\mathbb{D}} \text{ is complete } K\text{-spectral for } \varphi(T), \forall \varphi \in \Phi,$$
$$\implies \overline{X} \text{ is complete } K'\text{-spectral for } T$$

holds for every T with $\sigma(T) \subset X$.

We consider the cases $X = \Omega$ y $X = \overline{\Omega}$, where $\Omega \subset \mathbb{C}$ is a finitely connected domain with piecewise analytic boundary. The case $X = \overline{\Omega}$ (when $\sigma(T)$ can touch $\partial\Omega$) is technically more difficult.

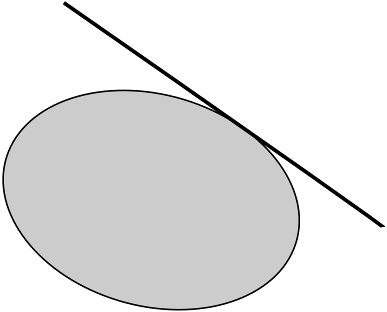
Results from the literature written in terms of test collections

- Let $\Omega_1, \ldots, \Omega_n \subset \widehat{\mathbb{C}}$ be simply connected domains with analytic boundaries which do not intersect and $\varphi_k : \overline{\Omega_k} \to \overline{\mathbb{D}}$ Riemann mappings. Then $\{\varphi_1, \ldots, \varphi_n\}$ is a test collection over $\overline{\bigcap \Omega_k}$. *(Douglas, Paulsen, 1986)*.



- Let $D_1, \ldots, D_n$ be discs in $\widehat{\mathbb{C}}$ and φ_k a Möbius transformation taking D_k onto $\mathbb{D}$. Then $\{\varphi_1, \ldots, \varphi_n\}$ is a test collection over $\overline{\bigcap D_k}$. *(Badea, Beckermann, Crouzeix, 2009)*.

- Let X be a convex compact. We write $X = \bigcap H_\alpha$, with H_α closed half-planes. Let φ_α be a Möbius transformation taking H_α onto $\overline{\mathbb{D}}$. Then $\{\varphi_\alpha\}$ is a test collection over X. *(Delyon, Delyon, 1999)*.



- If B is a finite Blaschke product,

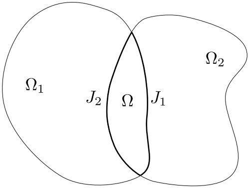
$$B(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}, \qquad |\lambda| = 1, \{a_j\}_{j=1}^{n} \subset \mathbb{D}.$$

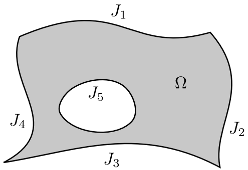
then the set $\{B\}$ is a test collection over $\overline{\mathbb{D}}$. *(Mascioni, 1994)*.

Admissible domains and functions

Definition

- $\Omega \subset \mathbb{C}$ a domain such that $\partial\Omega$ is a finite disjoint union of piecewise analytic Jordan curves. We assume that the interior angles of the “corners” of $\partial\Omega$ are in $(0, \pi]$.
- $\{J_k\}_{k=1}^n$ closed analytic arcs which intersect at most in their endpoints and such that $\partial\Omega = \bigcup J_k$.
- $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ with $|\varphi_k| = 1$ in J_k.
- Φ analytic in a neighbourhood of $\overline{\Omega}$ (can be weakened in many cases).
- φ_k' does not vanish in J_k.
- $\varphi_k(\zeta) \neq \varphi_k(z)$ if $\zeta \in J_k$, $z \in \overline{\Omega}$ and $z \neq \zeta$.





A simple example of admissible function

Example

$\Omega_1, \ldots, \Omega_n$ simply connected domains with analytic boundaries that intersect transversally.
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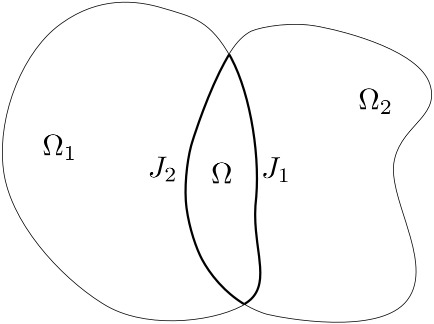
$\varphi_k : \overline{\Omega_k} \to \overline{\mathbb{D}}$ Riemann mappings.



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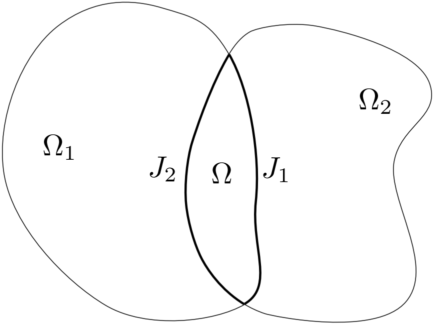

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Main results about admissible functions

Theorem

Let Ω be simply connected and $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ admissible. Then Φ is a test collection over $\overline{\Omega}$ (with constant K' depending on $\|T\|$). If moreover Φ is injective in $\overline{\Omega}$ and Φ' does not vanish in Ω, then the constant is independent of T.

Theorem

Let Ω be finitely connected and $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ admissible. Then Φ is a test collection over Ω (with constant K' depending on the value of $\|(T - \lambda I)^{-1}\|$ in a finite number of points). If moreover Φ is injective in $\overline{\Omega}$ and Φ' does not vanish in Ω, then the constant is independent of T.

In the first theorem $\sigma(T)$ may intersect $\partial\Omega$ while in the second theorem it may not. The case when $\sigma(T)$ intersects $\partial\Omega$ is technically much more difficult.

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Von Neumann's inequality

If T is a contraction, then

$$\|p(T)\| \leq \sup_{|z|\leq 1} |p(z)|$$

for every polynomial p (von Neumann's inequality).

If T_1, T_2 are commuting contractions, then

$$\|p(T_1, T_2)\| \leq \sup_{|z_j|\leq 1} |p(z_1, z_2)|$$

for every polynomial p in two variables (Ando).

However, for three or more commuting contractions $T_1, \ldots, T_n$, it is false in general that

$$\|p(T_1, \ldots, T_n)\| \leq \sup_{|z_j|\leq 1} |p(z_1, \ldots, z_n)|.$$

Open problem: It is unknown if there is a finite constant C_n such that

$$\|p(T_1, \ldots, T_n)\| \leq C_n \sup_{|z_j|\leq 1} |p(z_1, \ldots, z_n)|$$

for every polynomial p and commuting contractions $T_1, \ldots, T_n$.

This problem is important in the theory of several operators. It is believed that there is no such finite constant C_n.

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We denote by $\mathscr{B}$ the set of all tuples $\Phi = (\varphi_1, \ldots, \varphi_n)$ where φ_k are finite Blaschke products such that Φ is injective in $\overline{\mathbb{D}}$ and Φ' does not vanish in $\mathbb{D}$.

Theorem

If $n \geq 3$ and $p \in \mathbb{C}[z_1, \ldots, z_n]$,

$$\sup_{(T_1,\ldots,T_n):\|T_j\|\leq 1} \|p(T_1, \ldots, T_n)\| = \sup \|p(\varphi_1(T), \ldots, \varphi_n(T))\|,$$

where $\Phi = (\varphi_1, \ldots, \varphi_n)$ runs over all the tuples in $\mathscr{B}$ and T runs over all diagonalizable matrices such that $\sigma(T) \subset \mathbb{D}$ and $\|\varphi_k(T)\| \leq 1$, $k = 1, \ldots, n$.

We use a theorem of Agler, McCarthy and Young (2013) which says that it is enough to study von Neumann's inequality for contractions which are matrices with all their eigenvalues different (generic matrices). We also use Pick's interpolation problem to construct the Blaschke products (solving a problem whose data has been perturbed in an adecquate manner).

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Blaschke products and von Neumann's inequality

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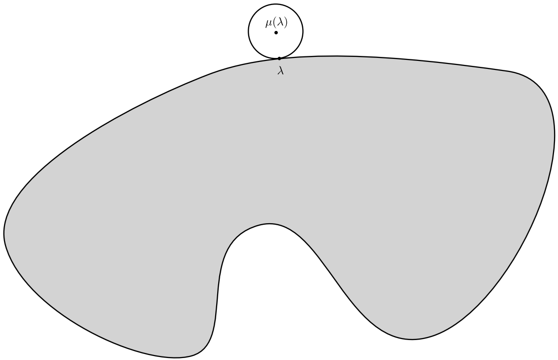
Theorem

Let $\Omega_1, \ldots, \Omega_n$ be Jordan domains whose boundaries are rectifiable, Ahlfors regular and intersect transversally. If $\overline{\Omega}_j$ is (complete) K_j-spectral for T, for $j = 1, \ldots, n$, then $\overline{\bigcap \Omega_j}$ is (complete) K-spectral for T.

This theorem generalizes the result of Badea, Beckermann, Crouzeix (2009) about the intersection of discs in the Riemann sphere.

Theorem

Let Ω be a piecewise C^2 Jordan domain and $R > 0$ such that for each $\lambda \in \Omega$ there exists a $\mu(\lambda) \in \mathbb{C} \setminus \overline{\Omega}$ such that $B(\mu, R)$ is tangent to $\partial\Omega$ at λ. If $\|(T - \mu(\lambda)I)^{-1}\| \leq R^{-1}$ for every $\lambda \in \partial\Omega$, then $\overline{\Omega}$ is complete K-spectral for T.



This theorem generalizes results of Delyon, Delyon (1999) and Putinar, Sandberg (2005) about convex sets which contain the numerical range of an operator. It can also be seen as a generalization of ρ-contractions.

Separation of singularities with the composition

- To prove our theorems about K-spectral sets it would be enough to decompose every $f \in A(\overline{\Omega})$ as

$$f(z) = g_1(\varphi_1(z)) + \cdots + g_n(\varphi_n(z)), \qquad g_k \in A(\overline{\mathbb{D}}). \tag{$*$}$$

- If φ_k are univalent in Ω_k, this is equivalent to writing

$$f(z) = f_1(z) + \cdots + f_n(z), \qquad f_k \in A(\overline{\Omega}_k).$$

- This is a separation of singularities (Havin, Nersessian, Ortega-Cerdà).
- In the general case, it is not possible to get $(*)$.

Theorem

If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, there are bounded linear operators $F_k : H^\infty(\Omega) \to H^\infty(\Omega)$ such that the operator

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Techniques of the proof

- Write
$$f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{1}{w-z} f(w)\,dw = \sum_k \frac{1}{2\pi i}\int_{J_k} \frac{1}{w-z} f(w)\,dw$$
- The integral operator
$$f \mapsto \int_{J_k} \left[\frac{1}{w-z} - \frac{\varphi_k'(w)}{\varphi_k(w)-\varphi_k(z)}\right] f(w)\,dw$$
is weakly singular and hence compact.
- Replace the Cauchy integrals
$$\int_{J_k} \frac{1}{w-z} f(w)\,dw$$
by *modified* Cauchy integrals
$$\int_{J_k} \frac{\varphi_k'(w)}{\varphi_k(w)-\zeta} f(w)\,dw, \qquad \zeta := \varphi_k(z).$$
- Use the trick of Havin-Nersessian to obtain functions of class H^∞ when we decompose f into a sum of Cauchy integrals over the arcs J_k.

Subalgebras $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$

$$\mathcal{H}_\Phi = \left\{ \sum_{j=1}^{l} f_{j,1}(\varphi_1(z)) f_{j,2}(\varphi_2(z)) \cdots f_{j,n}(\varphi_n(z)) : \ l \in \mathbb{N},\ f_{j,k} \in H^\infty(\mathbb{D}) \right\}$$

$$\mathcal{A}_\Phi = \left\{ \sum_{j=1}^{l} f_{j,1}(\varphi_1(z)) f_{j,2}(\varphi_2(z)) \cdots f_{j,n}(\varphi_n(z)) : \ l \in \mathbb{N},\ f_{j,k} \in A(\overline{\mathbb{D}}) \right\}$$

They are subalgebras (not closed, a priori) of $H^\infty(\Omega)$ and $A(\overline{\Omega})$ respectively.

Questions:

- What geometric conditions on Φ guarantee that $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$?
- What geometric conditions on Φ guarantee that $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$ are closed (or weak*-closed) subalgebras of finite codimension in $H^\infty(\Omega)$ and $A(\overline{\Omega})$ respectively?

Main results about $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$

Theorem

If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, then $\mathcal{H}_\Phi$ is a weak*-closed finite-codimensional subalgebra in $H^\infty(\Omega)$ and $\mathcal{A}_\Phi$ is a closed finite-codimensional subalgebra in $A(\overline{\Omega})$. If moreover Φ is injective in $\overline{\Omega}$ and Φ' does not vanish in Ω, then $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$.

Remark: It is easy to see that to obtain the equalities it is necessary that Φ is inyective and Φ' does not vanish.

This is a result about generation of algebras. Related problems have been studied by Wermer (1958), Bishop (1958), Blumenthal (1974), Sibony and Wermer (1974), Stessin and Thomas (2003), Matheson and Stessin (2005), and others.

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Compact operators: The operator

$$L(f) = \sum_k F_k(f) \circ \varphi_k$$

satisfies that $L - I$ is compact (previous theorem). Therefore the range L is closed and finite-codimensional. The range of L is contained in our subalgebra.

Banach algebras: Classfication of closed unital subalgebras of codimension one in a commutative Banach algebra (Gorin, 1969):

A subalgebra of codimension one has one of the two following forms:

- $\{f : f(a) = f(b)\}$
- $\{f : f'(a) = 0\}$

(we identify elements in the algebra with functions using Gelfand's transform and we consider pointwise derivations in the algebraic sense).

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Outline

1. Part I: Test functions and K-spectral sets
   - Test collections
   - Subalgebras of analytic functions
   - An application: operators with spectra on a curve

2. Part II: Separating structures and operator tuples
   - Separating structures
   - Vessels and generalized compression

Resolvent estimates and similarity to a normal

- T an operator with $\sigma(T) \subset \Gamma$, where Γ is a smooth curve without self-intersections.
- It holds that $\|(T - \lambda I)^{-1}\| \leq \operatorname{dist}(\lambda, \Gamma)^{-1}$ for every λ in a neighbourhood of Γ if and only if T is normal (Stampfli, 1965).
- If T is similar to a normal (VTV^{-1} is normal), then $\|(T - \lambda I)^{-1}\| \leq C \operatorname{dist}(\lambda, \Gamma)^{-1}$. The converse is false: Markus (1964), Benamara-Nikolski (1999), Nikolski-Treil (2002).

Theorem

If Ω is a $C^{1+\alpha}$ Jordan domain, $\Gamma = \partial\Omega$, U is a neighbourhood of Γ and

$$\|(T - \lambda I)^{-1}\| \leq \operatorname{dist}(\lambda, \Gamma)^{-1}, \qquad \lambda \in U \setminus \overline{\Omega},$$
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The two growth conditions can be interchanged.

In the proof of this theorem we use our generalization of the theorem of Delyon and Delyon to show that $\overline{\Omega}$ is K-spectral for T.

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Generalization of a theorem of van Casteren

Ω a $C^{1+\alpha}$ Jordan domain, $\Gamma = \partial\Omega$. $\{\gamma_s\}_{0<s<1}$ is a family of curves that *tends nicely to* Γ (when $s \to 0$) if:

1. $C^{-1}s \leq \operatorname{dist}(x, \Gamma) \leq Cs, \quad x \in \gamma_s$
2. $\operatorname{long}(\gamma_s \cap B(x, r)) \leq Cr$

If $\gamma_s \subset \mathbb{C} \setminus \overline{\Omega}$ for every s, we say that $\{\gamma_s\}$ *tends nicely to* Γ *from the outside.*

Theorem (Generalization of van Casteren)

If $\|(T - \lambda I)^{-1}\| \leq C \operatorname{dist}(\lambda, \Gamma)^{-1}$ *for every* $\lambda \in \Omega$ *and*

$$\int_{\gamma_s} \|(T - \lambda I)^{-1}x\|^2 \, |d\lambda| \leq C\|x\|^2 s^{-1},$$

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for some family of curves $\{\gamma_s\}_{0<s<1}$ *which tends nicely to* Γ *from the outside, then* T *is similar to a normal.*

The case $\Omega = \mathbb{D}$, $\gamma_s = \{z \in \mathbb{C} : |z| = 1 + s\}$ is due to van Casteren (1984).

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Tools

Pseudoanalytic extension: If $f \in C^{1+\alpha}(\Gamma)$, then there exists $F \in C^1(\mathbb{C})$ such that $F|\Gamma = f$ and $|\overline{\partial} F(z)| \leq C\|f\|_{C^{1+\alpha}} \operatorname{dist}(z,\Gamma)^\alpha$.

Dynikin's functional calculus: If $\|(T-\lambda I)^{-1}\| \leq C \operatorname{dist}(\lambda,\Gamma)^{-1}$, we can define $f(T)$ for $f \in C^{1+\alpha}(\Gamma)$ by

$$f(T) = \frac{1}{2\pi i}\int_{\partial D} F(\lambda)(\lambda I - T)^{-1}\,d\lambda - \frac{1}{\pi}\iint_D \overline{\partial} F(\lambda)(\lambda I - T)^{-1}\,dA(\lambda),$$

where F is any pseudoanalytic extension of f and $D \supset \Gamma$ is a domain.

Outline

Definition

A *separating structure* is a Hilbert space K, a pair of commuting selfadjoint operators $A_1, A_2 \in \mathcal{B}(K)$ and an orthogonal decomposition

$$K = \overbrace{H_{0,-} \oplus M_-}^{H_-} \oplus \overbrace{M_+ \oplus H_{0,+}}^{H_+},$$

with $\dim M_- = \dim M_+ < \infty$ such that A_1, A_2 have the structure

$$A_j = \begin{bmatrix} * & * & 0 & 0 \\ * & \Lambda_{-1} & R_{-1} & 0 \\ 0 & T_0 & \Lambda_0 & * \\ 0 & 0 & * & * \end{bmatrix}, \qquad j = 1, 2.$$

Usually we work with the normal operator $N = A_1 + iA_2$ instead of the pair A_1, A_2.

The finite dimensional space $M = M_- \oplus M_+$ is used to characterize the behaviour of the separating structure using some auxiliary matrices (which are built using $\Lambda_{-1}, \Lambda_0, R_{-1}, T_0$).

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An example: subnormal operators of finite type (following Xia and Yakubovich)

$S \in \mathcal{B}(H)$ is subnormal if $S = N|H$, with $N \in \mathcal{B}(K)$ normal, $K \supset H$.
S is pure subnormal if there is no non-trivial subspace H_0 which reduces S and such that $S|H_0$ is normal.

If S is pure subnormal, it has a minimal normal extension: $N = \begin{bmatrix} S'^* & 0 \\ X & S \end{bmatrix}$.

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If S is pure subnormal of finite type,

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$A_1 = \operatorname{Re}(aN + bN^*)$, $A_2 = \operatorname{Im}(aN + bN^*)$ form a separating structure ($a, b \in \mathbb{C}$).

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We define auxiliary matrices $\alpha, \gamma \in \mathcal{B}(M)$ by

$$\alpha = \begin{bmatrix} 0 & -R_{-1} \\ T_0 & 0 \end{bmatrix}, \qquad s = P_M N|M = \begin{bmatrix} \Lambda_{-1} & R_{-1} \\ T_0 & \Lambda_0 \end{bmatrix}, \qquad \gamma = -(\alpha^* s + \alpha s^*).$$

Then

$$\alpha^* P_M N + \alpha P_M N^* + \gamma = 0.$$

This motivates the definition of the *discriminant curve*:

$$X = \{(z, w) \in \mathbb{C}^2 : \det(z\alpha^* + w\alpha + \gamma) = 0\}.$$

We have $\sigma(N) \subset \{z \in \mathbb{C} : (z, \overline{z}) \in X\}$.

Mosaic function

It is a generalization of the mosaic function defined by Xia for subnormal operators.

Definition

The mosaic function ν is

$$\nu(z) = P_M(N - z)^{-1} P_{H_+}(N - z)|M, \qquad z \notin \sigma(N).$$

Its values are parallel projections in M.

The auxiliary matrices α, γ and the mosaic function ν contain all the information about the separating structure.
If $m_1, m_2 \in M$ y $z, w \notin \sigma(N)$, then

$$\langle (N^* - \overline{w})^{-1} m_1, (N^* - \overline{z})^{-1} m_2 \rangle = \langle (\gamma + z\alpha^* + w\alpha)^{-1} (I - \alpha\nu(z)\alpha^{-1} - \nu(w)^*) m_1, m_2 \rangle.$$

In non-degenerate cases, the linear span of the vectors $(N^* - \overline{z})^{-1} m$, with $m \in M$, $z \notin \sigma(N)$, is dense in K. Hence, we can recover the scalar product in K.

Question: Is it possible to recover the mosaic function ν from the matrices α and γ? In this case, the separating structure (which contains infinite-dimensional objects) comes determined by a finite-dimensional amount of data.

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The meromorphic function Q

We put

$$\Sigma = -\alpha^{-1}\alpha^*, \qquad D = -\alpha^{-1}\gamma.$$

The equation of the curve X rewrites as

$$\det(z\Sigma + D - wI) = 0.$$

If $p = (z, w) \in X$, we put

$$Q(p) = \Pi_w(z\Sigma + D) = \frac{1}{2\pi i}\int_{|\lambda - w| = \varepsilon} (z\Sigma + D - \lambda I)^{-1}\, d\lambda$$

the Riesz projection of $z\Sigma + D$ associated with the eigenvalue w.
Then $Q(p)$ is a parallel projection in M and for every z_0

$$\sum_{p=(z_0,w)\in X} Q(p) = I_M.$$

Q is a meromorphic function on X (we understand X as a finite union of Riemann surfaces, using the blow up).

Restoration formula

The algebraic curve X is a real curve and it comes equipped with a complex conjugation. We say that it is *separated* if its real part $X_{\mathbb{R}}$ divides the curve into two halves (which are exchanged by the complex conjugation).

The matrix Σ has the form

$$\Sigma = \begin{bmatrix} \Sigma^- & 0 \\ 0 & \Sigma^+ \end{bmatrix}.$$

Theorem (Restoration formula)

If $\sigma(\Sigma^-) \cap \sigma(\Sigma^+) = \emptyset$, then the discriminant curve X is separated, $X = X_- \cup X_{\mathbb{R}} \cup X_+$, and for every z_0,

$$\nu(z_0) = \sum_{p=(z_0,w)\in X_+} Q(p).$$

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$$\nu(z_0) = \sum_{p=(z_0,w)\in X_+} Q(p).$$

Outline

Vessels

A theory developed by Livšic, Vinnikov and others.
We start with two commuting operators $B_1, B_2 \in \mathcal{B}(H)$ which have finite-dimensional imaginary part. We put

$$M = (B_1 - B_1^*)H + (B_2 - B_2^*)H.$$

We define selfadjoint auxiliary matrices in M:

$$\sigma_j = \frac{1}{i}(B_j - B_j^*)|M, \qquad j = 1, 2,$$

and γ^{in}, γ^{out}.

We define the *discriminant polynomial*

$$\Delta(x_1, x_2) = \det(x_1\sigma_2 - x_2\sigma_1 + \gamma^{\text{in}}) = \det(x_1\sigma_2 - x_2\sigma_1 + \gamma^{\text{out}})$$

and the *discriminant curve*

$$X = \{(x_1, x_2) \in \mathbb{C}^2 : \Delta(x_1, x_2) = 0\}.$$

We have $\sigma(B_1, B_2) \subset X$.

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Recall: Compression and dilation

If $A \in \mathcal{B}(K)$, K decomposes as

$$K = G_- \oplus H \oplus G_+$$

and A has the structure

$$A = \begin{bmatrix} * & 0 & 0 \\ * & B & 0 \\ * & * & * \end{bmatrix},$$

then B is called a *compression* of A and A is called a *dilation* of B.

For every polynomial p,

$$p(A) = \begin{bmatrix} * & 0 & 0 \\ * & p(B) & 0 \\ * & * & * \end{bmatrix}.$$

Every contraction can be dilated to a unitary. Using this one can prove von Neumann's inequality in a simple manner.

Remark: G_+ and $H \oplus G_+$ are invariant for A.

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Single operator

Spectral theory of contractions of Sz.-Nagy and Foias.

$B \in \mathcal{B}(H)$, $\|B\| \leq 1$

$$\text{dilation} \Downarrow \quad \Uparrow \text{compression}$$

Spectral theory of isometries. Functional model over $H^2(\mathbb{D})$ (vector valued).

A isometry/unitary.

Tuples of operators

Livšic-Vinnikov theory.

B_1, B_2, $\dim \operatorname{Im} B_j < \infty$

$$\text{dilation} \downarrow \quad \Uparrow \text{compression}$$

Separating structures. (Functional model over H^2 of the halves of the discriminant curve).

A_1, A_2 selfadjoint.

Compression of separating structures

$A_1, A_2 : K \to K$ selfadjoint. Two separating structures for A_1, A_2

$$\omega : K = \overbrace{H_{0,-} \oplus M_-}^{H_-} \oplus \overbrace{M_+ \oplus H_{0,+}}^{H_+}, \qquad \widehat{\omega} : K = \overbrace{\widehat{H}_{0,-} \oplus \widehat{M}_-}^{\widehat{H}_-} \oplus \overbrace{\widehat{M}_+ \oplus \widehat{H}_{0,+}}^{\widehat{H}_+}.$$

We assume that these structures are subordinate ($\widehat{\omega} \prec \omega$), which means that

$$\widehat{H}_+ \subset H_+ \qquad (\Longleftrightarrow H_- \subset \widehat{H}_-).$$

This is a partial order relation.

We define a notion of *generalized compression* (where no subspace is required to be invariant).

Lemma

The operators A_1, A_2 can be compressed to $H_+/\widehat{H}_+$ if and only if the matrix $P_{M_+}|\widehat{M}_+ : \widehat{M}_+ \to M_+$ is invertible.

Theorem

If B_1, B_2 are the generalized compressions of A_1 and A_2 to $H_+/\widehat{H}_+$, then B_1, B_2 form a vessel and the auxiliary matrices of the vessel $\sigma_1, \sigma_2, \gamma^{in}, \gamma^{out}$ can be written in terms of the auxiliary matrices of the separating structure α, γ in a simple way. In particular, the discriminant curves of the vessel and separating structure coincide.

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List of articles

- M. A. Dritschel, D. Estévez and D. Yakubovich, *Traces of analytic uniform algebras on subvarieties and test collections*, J. London Math. Soc. **95** (2017), no. 2, 414–440.
- M. A. Dritschel, D. Estévez and D. Yakubovich, *Tests for complete K-spectral sets*, accepted in J. Funct. Anal.; preprint available at `arXiv:15010.08350`.
- M. A. Dritschel, D. Estévez and D. Yakubovich, *Linear resolvent growth for operators with spectra on a curve and their similarity to normals*, preprint available at `arXiv:1704.08135`.