Explicit traces of functions on Sobolev spaces and quasi-optimal linear interpolators

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Definition of Trace Spaces for RKHSs

- H a Reproducing Kernel Hilbert Space (RKHS) of functions on Ω.
- $\Lambda = \{\lambda_n\}_{n=0}^{\infty} \subset \Omega$ a sequence of distinct points.
- The *trace space*: $H|_\Lambda = \{F|_\Lambda : F \in H\}$. It is a sequence space.
- The *trace norm*: $\|f\|_{H|_\Lambda} = \inf\{\|F\|_H : F|_\Lambda = f\}$. It makes $H|_\Lambda$ a Hilbert space.
- $H|_\Lambda \cong \overline{\operatorname{span}\{k_{\lambda_n}\}}$, where k_λ is the reproducing kernel at λ.
- **Problem:** Characterize the trace space.

Some Results for Spaces of Analytic Functions

- L. Carleson, H.S. Shapiro, A.L. Shields (1961), $H^p(\mathbb{D})|_\Lambda = L^p(\sum_{n\in\mathbb{N}}(1-|\lambda_n|^2)d\delta_{\lambda_n})$, Λ a Carleson sequence.
- J. Bruna, A. Nicolau, K. Øyma (1996), $H^p(\mathbb{D})|_\Lambda$, for Λ non-Carleson.
- A.P. Schuster, K. Seip (1998), Bergman spaces $A^p(\mathbb{D})$.
- A. Hartmann (2001), weighted Bergman spaces $B^{p,\alpha}(\mathbb{D})$.
- N. Arcozzi, R. Rochberg, E. Sawyer (2008), Besov-Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$.

Definition of Sobolev Spaces

- $W^{r,p}(\mathbb{R}^d)$ the non-homogeneous Sobolev space:

$$\|F\|^p_{W^{r,p}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (|F(x)|^p + |D^r F(x)|^p)\, dx.$$

- $L^{r,p}(\mathbb{R}^d)$ the homogeneous Sobolev space:

$$\|F\|^p_{L^{r,p}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |D^r F(x)|^p\, dx.$$

- Sobolev embedding theorem:
  1. If $p > d$, then $W^{r,p}(\mathbb{R}^d) \subset \mathcal{C}^{r-1,\alpha}(\mathbb{R}^d)$, with $\alpha = 1 - \frac{d}{p}$.
  2. If $rp > d$, then $W^{r,p}(\mathbb{R}^d) \subset \mathcal{C}(\mathbb{R}^d)$.
- $W^{r,2}(\mathbb{R}^d)$ is a RKHS if $2r > d$.
- The trace space and trace norm can also be considered for the spaces $W^{r,p}(\mathbb{R}^d)$, $L^{r,p}(\mathbb{R}^d)$, with $rp > d$.

Extension Problems for Sobolev Spaces

- Put $\mathbb{X} = W^{r,p}(\mathbb{R}^d)$ or $L^{r,p}(\mathbb{R}^d)$, with $rp > d$. Fix $E \subset \mathbb{R}^d$.
- The trace space: $\mathbb{X}|_E = \{F|_E : F \in \mathbb{X}\}$.
- The trace norm: $\|f\|_{\mathbb{X}|_E} = \inf\{\|F\|_{\mathbb{X}} : F|_E = f\}$.
- **Problem 1:** Describe $\mathbb{X}|_E$. When does $f : E \to \mathbb{R}$ extend to an $F \in \mathbb{X}$ with $F|_E = f$? Is there a linear extension operator $T : \mathbb{X}|_E \to \mathbb{X}$ such that $(Tf)|_E = f$?
- **Problem 2:** If $f \in \mathbb{X}|_E$, find an $F \in \mathbb{X}$ with $F|_E = f$ and $\|F\|_{\mathbb{X}} \leq C\|f\|_{\mathbb{X}|_E}$? Is there a bounded linear extension operator (BLEO) $T : \mathbb{X}|_E \to \mathbb{X}$ (bounded and satisfying $(Tf)|_E = f$)?
- **Problem 3:** Find a formula for an equivalent norm in $\mathbb{X}|_E$.
- If $\mathbb{X}$ is Hilbert and $J : \mathbb{X} \to \mathbb{X}|_E$ is the restriction, J^* is a BLEO.
- However, we want to find simple and explicit formulas.

The Problem We Solve

- We assume $d = 1$, $E = \Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$, with $\lambda_n < \lambda_{n+1}$, $1 < p < \infty$.
- We define $I = \bigcup_{n\in\mathbb{Z}}[\lambda_n, \lambda_{n+1}]$. We work in $W^{r,p}(I)$ or $L^{r,p}(I)$.
- When working in $W^{r,p}(I)$, we assume that

$$h_n = \lambda_{n+1} - \lambda_n \leq K, \qquad n \in \mathbb{Z}.$$

- Main results for $r = 1, 2$.
- Can be seen as an interpolation problem: If T is a BLEO, then the function Tf interpolates the data f, and $\|Tf\|$ is optimal up to a constant factor. We call such a T a quasi-optimal interpolator.

Definition of Divided Differences

- $f(x_1, x_2) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.
- $f(x_1, \dots, x_k) = \dfrac{f(x_2, \dots, x_k) - f(x_1, \dots, x_{k-1})}{x_k - x_1}, \qquad k \geq 3.$
- Mean value theorem: If $F \in \mathcal{C}^{k-1}$, there is $\xi \in [\min x_j, \max x_j]$ such that $F^{(k-1)}(\xi) = F(x_1, \dots, x_k)$.

The Equivalent Norm

$$\|f\|_{\mathrm{eq},L}^p = \sum_{n\in\mathbb{Z}} (\lambda_{n+r} - \lambda_n) |f(\lambda_n, \dots, \lambda_{n+r})|^p.$$

$$\|f\|_{\mathrm{eq},W}^p = \|f\|_{\mathrm{eq},L}^p + \sum_{j=0}^{r-1} \sum_{n\in\mathbb{Z}} (\lambda_{n+r} - \lambda_n)^{jp+1} |f(\lambda_n, \dots, \lambda_{n+j})|^p.$$

Theorem

If $r = 1, 2$, *then* $c(r,p)\|f\|_{eq,L} \leq \|f\|_{L^{r,p}(I)|_\Lambda} \leq C(r,p)\|f\|_{eq,L}$, *and* $c(r,p,K)\|f\|_{eq,W} \leq \|f\|_{W^{r,p}(I)|_\Lambda} \leq C(r,p,K)\|f\|_{eq,W}$.
Moreover, for $r = 2$,

$$\|f\|_{eq,W}^p \approx \|f\|_{eq,L}^p + \sum_{n\in\mathbb{Z}} (\lambda_{n+1} - \lambda_{n-1}) |f(\lambda_n)|^p.$$

Quasi-Optimal Spline Interpolators

Theorem

For $r = 1, 2$, there are BLEO Φ_r with $\|\Phi_r\|_{L^{r,p}(I)|_E \to L^{r,p}(I)} \leq C(r,p)$ and $\|\Phi_r\|_{W^{r,p}(I)|_\Lambda \to W^{r,p}(I)} \leq C(r,p,K)$. Moreover, $\Phi_r f$ is a piecewise polynomial for any f.

- Φ_1 is the piecewise linear interpolator:

$$(\Phi_1 f)(x) = f(\lambda_n)\frac{\lambda_{n+1} - x}{h_n} + f(\lambda_{n+1})\frac{x - \lambda_n}{h_n}, \qquad \lambda_n \leq x \leq \lambda_{n+1}.$$

  Recall that $h_n = \lambda_{n+1} - \lambda_n$.
- Φ_2 is a cubic spline.

Definition of Φ_2

- We define the auxiliary interpolation nodes

$$\mu_n = \frac{\lambda_n + \lambda_{n+1}}{2}.$$

- We define the averages of slopes

$$\alpha_n(f) = \frac{h_n f(\lambda_{n-1}, \lambda_n) + h_{n-1} f(\lambda_n, \lambda_{n+1})}{h_{n-1} + h_n}.$$

- Conditions for the cubic spline in the interval $[\lambda_n, \mu_n]$:

$$(\Phi_2 f)(\lambda_n) = f(\lambda_n), \qquad (\Phi_2 f)'(\lambda_n) = \alpha_n(f),$$

$$(\Phi_2 f)(\mu_n) = \frac{f(\lambda_n) + f(\lambda_{n+1})}{2}, \qquad (\Phi_2 f)'(\mu_n) = f(\lambda_n, \lambda_{n+1}).$$

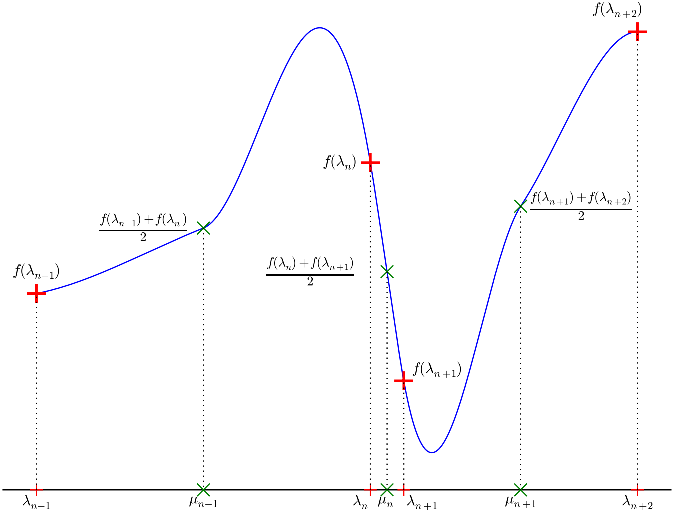
Explicit Formula for Φ_2

- The polynomial $q(x) = 4(x^2 - x^3)$ satisfies $q(0) = q'(0) = 0$ y $q(\frac{1}{2}) = \frac{1}{2}$, $q'(\frac{1}{2}) = 1$.
- Formula for $\lambda_n \leq x \leq \mu_n$:

$$f(\lambda_n) + \alpha_n(f)(x - \lambda_n) + h_n^2 f(\lambda_{n-1}, \lambda_n, \lambda_{n+1}) q\left(\frac{x - \lambda_n}{h_n}\right).$$

- Similar formula for $\mu_{n-1} \leq x \leq \lambda_n$.

Graph of the Interpolator Φ_2



Lemma

For any $r \geq 1$, we have

$$\|F\|_{W^{r,p}(I)}^p \approx \|F\|_{L^{r,p}(I)}^p + \sum_{j=0}^{r-1} \sum_{n\in\mathbb{Z}} (\lambda_{n+r} - \lambda_n)^{jp+1} |F(\lambda_n, \dots, \lambda_{n+j})|^p.$$

Idea of the Proof of the Results

1. If $f = F|_\Lambda$, then $\|F\|_{L^{r,p}(I)} \geq c(r,p)\|f\|_{\mathrm{eq},L}$, for $r \geq 1$.
2. $\|\Phi_r f\|_{L^{r,p}(I)} \leq C(r,p)\|f\|_{\mathrm{eq},L}$, for $r = 1, 2$.
3. This shows that $\|f\|_{\mathrm{eq},L} \approx \|f\|_{L^{r,p}(I)|_\Lambda}$ and Φ_r is bounded.
4. Use the Lemma to pass to $W^{r,p}$.

Some Related Works

- G.K. Luli (2008), preprint. Interpolator for $L^{r,p}(\mathbb{R})$, $r \geq 1$. Pastes interpolating polynomials on $r+1$ points using partitions of unity.
- P.A. Shvartsman (2009). Extension for $W^{1,p}(\mathbb{R}^d)$, $d \geq 1$.
- A. Israel (2010). Extension for $L^{2,p}(\mathbb{R}^2)$, $p > 2$.

Theorem (C. Fefferman, A. Israel, G.K. Luli; 2012)

If $p > n$, $r \geq 1$, $d \geq 1$, there is a BLEO $T : L^{r,p}(\mathbb{R}^d)|_E \to L^{r,p}(\mathbb{R}^d)$ with $\|T\| \leq C(r,p,d)$.

Bounded Depth Operators

Definition

An extension operator $T : \mathbb{X}|_E \to \mathbb{X}$ has depth smaller or equal than D if

$$(Tf)(x) = \sum_{y \in E} \phi(x,y) f(y)$$

and

$$\#\{y : \phi(x,y) \neq 0\} \leq D, \qquad \forall x.$$

- Our interpolators Φ_1 and Φ_2 have depth 2 and 3 respectively.
- The interpolators of G.K. Luli have also bounded depth $4(k-1)$.
- In general, for $d \geq 2$ one cannot hope to have bounded depth.

No Bounded Depth For $d \geq 2$

Theorem (C. Fefferman, A. Israel, G.K. Luli; 2012)

If $p > 2$, $A \geq 1$, $D \geq 1$, there exists a finite $E \subset \mathbb{R}^2$ such that E has no BLEO of norm smaller than A and depth smaller than D on $L^{2,p}(\mathbb{R}^2)$.

Open Questions

1. Is it true that $\|f\|_{\text{eq},L}$ and $\|f\|_{\text{eq},W}$ give equivalent norms for $r \geq 3$?
2. Can one give BLEOs with a simple formula for $r \geq 3$?
3. Can one generalize some of our results to $d \geq 2$ if one imposes some regularity conditions on the set of nodes Λ?